

The background of the slide is a solid blue color with a large, faint watermark of the University of Vienna seal. The seal features a central figure, likely a scholar or saint, seated and holding a book. Above the figure are three smaller figures in a row. The entire scene is enclosed within a circular border containing Latin text. The watermark is centered and serves as a background for the text.

Measure Theory for Probabilists

6. σ -additivity

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Proposition 2.8

- ▶ μ is σ -additive iff

$$\mu\left(\biguplus_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

- ▶ μ is σ -sub-additive iff

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

- ▶ μ is continuous from below, if for A, A_1, A_2, \dots and $A_1 \subseteq A_2 \subseteq \dots$ with $A = \bigcup_{n=1}^{\infty} A_n$,

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n).$$

- ▶ μ is continuous from above (in the \emptyset), if for $A(= \emptyset), A_1, A_2, \dots, \mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq \dots$ with $A = \bigcap_{n=1}^{\infty} A_n$,

$$(0 =) \mu(A) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Proposition 2.8

► Let \mathcal{R} be a ring and $\mu : \mathcal{R} \rightarrow \overline{\mathbb{R}}_+$ be additive and $\mu(A) < \infty$ for all $A \in \mathcal{R}$. Then, the following are equivalent:

1. μ is σ -additive;
2. μ is σ -subadditive;
3. μ is continuous from below;
4. μ is continuous from above in \emptyset ;
5. μ is continuous from above.

► Proof: 1. \Leftrightarrow 2., 5. \Rightarrow 4.: clear.

1. \Rightarrow 3.: With $A_0 = \emptyset$, $A = \bigcup_{n=1}^{\infty} A_n \setminus A_{n-1}$

3. \Rightarrow 1.: Set $A_N = \bigcup_{n=1}^N B_n$,

4. \Rightarrow 5.: With $B_n := A_n \setminus A \downarrow \emptyset$,
 $\mu(A_n) = \mu(B_n) + \mu(A) \xrightarrow{n \rightarrow \infty} \mu(A)$.

3. \Rightarrow 4.: Set $B_n := A_1 \setminus A_n \uparrow A_1$. Then,

$\mu(A_1) = \lim_{n \rightarrow \infty} \mu(B_n) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_n)$.

4. \Rightarrow 3. Set $B_n := A \setminus A_n \downarrow \emptyset$. Then,

$0 = \lim_{n \rightarrow \infty} \mu(B_n) = \mu(A) - \lim_{n \rightarrow \infty} \mu(A_n)$.

Inner regularity of measures on Polish spaces

- ▶ Lemma 2.9: (Ω, \mathcal{O}) Polish, μ finite, $\varepsilon > 0$.
There exists $K \subseteq \Omega$ compact with $\mu(\Omega \setminus K) < \varepsilon$.
- ▶ Proof: There is $\{\omega_1, \omega_2, \dots\} \subseteq \Omega$ dense, so
 $\Omega = \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)$. μ is continuous from above \Rightarrow

$$0 = \mu\left(\Omega \setminus \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)\right) = \lim_{N \rightarrow \infty} \mu\left(\Omega \setminus \bigcup_{k=1}^N B_{1/n}(\omega_k)\right).$$

Take $N_n \in \mathbb{N}$ with $\mu\left(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k^n)\right) < \varepsilon/2^n$ and

$A := \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)$ totally bounded, hence relatively compact with

$$\begin{aligned} \mu(\Omega \setminus \bar{A}) &\leq \mu(\Omega \setminus A) \leq \mu\left(\bigcup_{n=1}^{\infty} \left(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\right)\right) \\ &\leq \sum_{n=1}^{\infty} \mu\left(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\right) < \varepsilon. \end{aligned}$$

Inner regularity and σ -additivity

- ▶ Theorem 2.10: \mathcal{H} semi-ring, $\mu : \mathcal{H} \rightarrow \mathbb{R}_+$ finite, finitely additive and inner $\mathcal{K} \subseteq \mathcal{H}$ -regular. Then μ is σ -additive.
- ▶ Proof: Wlog, \mathcal{H} is ring and $\mathcal{K} = \mathcal{K}_\cup$

To show: μ is continuous from above in \emptyset . Let $A_1, A_2, \dots \in \mathcal{H}$ with $A_1 \supseteq A_2 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$ and $\varepsilon > 0$.

Choose $K_1, K_2, \dots \in \mathcal{K}$ with $K_n \subseteq A_n, n \in \mathbb{N}$ and

$$\mu(A_n) \leq \mu(K_n) + \varepsilon 2^{-n}.$$

Then, $\bigcap_{n=1}^{\infty} K_n \subseteq \bigcap_{n=1}^{\infty} A_n = \emptyset$, so there is $N \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n = \emptyset$. From this,

$$A_N = A_N \cap \left(\bigcup_{n=1}^N K_n^c \right) = \bigcup_{n=1}^N A_N \setminus K_n \subseteq \bigcup_{n=1}^N A_n \setminus K_n.$$

By subadditivity and monotonicity of μ , for $m \geq N$,

$$\mu(A_m) \leq \mu(A_N) \leq \sum_{n=1}^N \mu(A_n \setminus K_n) \leq \varepsilon \sum_{n=1}^N 2^{-n} \leq \varepsilon.$$