Measure Theory for Probabilists
6. σ-additivity

Peter Pfaffelhuber

January 6, 2024

Proposition 2.8

- μ is σ-additive iff

$$\mu\Big(\biguplus_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

- μ is σ-sub-additive iff

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

- μ is continuous from below, if for $A, A_1, A_2, \ldots$ and $A_1 \subseteq A_2 \subseteq \ldots$ with $A = \bigcup_{n=1}^{\infty} A_n$,

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

- μ is continuous from above (in the $\emptyset$), if for $A(=\emptyset), A_1, A_2, \ldots,\ \mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq \ldots$ with $A = \bigcap_{n=1}^{\infty} A_n$,

$$(0 =)\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

Proposition 2.8

- Let $\mathcal{R}$ be a ring and $\mu : \mathcal{R} \to \overline{\mathbb{R}}_+$ be additive and $\mu(A) < \infty$ for all $A \in \mathcal{R}$. Then, the following are equivalent:
  1. μ is σ-additive;
  2. μ is σ-subadditive;
  3. μ is continuous from below;
  4. μ is continuous from above in $\emptyset$;
  5. μ is continuous from above.
- Proof: 1.⇔2., 5.⇒4.: clear.
  1.⇒3.: With $A_0 = \emptyset$, $A = \biguplus_{n=1}^{\infty} A_n \setminus A_{n-1}$
  3.⇒1.: Set $A_N = \biguplus_{n=1}^{N} B_n$,
  4.⇒5.: With $B_n := A_n \setminus A \downarrow \emptyset$,
  $\mu(A_n) = \mu(B_n) + \mu(A) \xrightarrow{n\to\infty} \mu(A)$.
  3.⇒4.: Set $B_n := A_1 \setminus A_n \uparrow A_1$. Then,
  $\mu(A_1) = \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$.
  4.⇒3. Set $B_n := A \setminus A_n \downarrow \emptyset$. Then,
  $0 = \lim_{n\to\infty} \mu(B_n) = \mu(A) - \lim_{n\to\infty} \mu(A_n)$.

Inner regularity of measures on Polish spaces

- Lemma 2.9: $(\Omega, \mathcal{O})$ Polish, μ finite, $\varepsilon > 0$. There exists $K \subseteq \Omega$ compact with $\mu(\Omega \setminus K) < \varepsilon$.
- Proof: There is $\{\omega_1, \omega_2, \dots\} \subseteq \Omega$ dense, so $\Omega = \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)$. μ is continuous from above $\Rightarrow$

$$0 = \mu\Big(\Omega \setminus \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)\Big) = \lim_{N\to\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N} B_{1/n}(\omega_k)\Big).$$

Take $N_n \in \mathbb{N}$ with $\mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k^n)\Big) < \varepsilon/2^n$ and $A := \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)$ totally bounded, hence relatively compact with

$$\mu(\Omega \setminus \overline{A}) \leq \mu(\Omega \setminus A) \leq \mu\Big(\bigcup_{n=1}^{\infty}\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big)\Big)$$

$$\leq \sum_{n=1}^{\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big) < \varepsilon.$$

Inner regularity and σ-additivity

- Theorem 2.10: $\mathcal{H}$ semi-ring, $\mu : \mathcal{H} \to \mathbb{R}_+$ finite, finitely additive and inner $\mathcal{K} \subseteq \mathcal{H}$-regular. Then μ is σ-additive.
- Proof:Wlog, $\mathcal{H}$ is ring and $\mathcal{K} = \mathcal{K}_\cup$
  To show: μ is continuous from above in $\emptyset$. Let $A_1, A_2, \dots \in \mathcal{H}$ with $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigcap_{n=1}^\infty A_n = \emptyset$ and $\varepsilon > 0$.
  Choose $K_1, K_2, \dots \in \mathcal{K}$ with $K_n \subseteq A_n, n \in \mathbb{N}$ and

$$\mu(A_n) \leq \mu(K_n) + \varepsilon 2^{-n}.$$

  Then, $\bigcap_{n=1}^\infty K_n \subseteq \bigcap_{n=1}^\infty A_n = \emptyset$, so there is $N \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n = \emptyset$. From this,

$$A_N = A_N \cap \left(\bigcup_{n=1}^N K_n^c \right) = \bigcup_{n=1}^N A_N \setminus K_n \subseteq \bigcup_{n=1}^N A_n \setminus K_n.$$

  By subadditivity and monotonicity of μ, for $m \geq N$,

$$\mu(A_m) \leq \mu(A_N) \leq \sum_{n=1}^N \mu(A_n \setminus K_n) \leq \varepsilon \sum_{n=1}^N 2^{-n} \leq \varepsilon.$$