Measure Theory for Probabilists
5. Set functions and outer measures

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Definition 2.1

- For $\mathcal{F} \subseteq 2^\Omega$, we call $\mu : \mathcal{F} \to \overline{\mathbb{R}}_+$ a set function.
- μ is *finitely additive* if

$$\mu\Big(\biguplus_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \mu(A_k).$$

  for disjoint $A_1, \ldots, A_n \in \mathcal{F}$.
- $\mu : \mathcal{F} \to \mathbb{R}_+$ is *σ-additive* if the same holds for $n = \infty$.
- If $\mathcal{F}$ is a σ-algebra, and μ is σ-additive, μ is a *measure* and $(\Omega, \mathcal{F}, \mu)$ is a *measure space*.
- If $\mu(\Omega) < \infty$, then μ is a *finite measure*; if $\mu(\Omega) = 1$, μ is a *probability measure*. Then, $(\Omega, \mathcal{F}, \mu)$ is a *probability space*.

Definition 2.1

- μ is called *sub-additive* if

$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) \leq \sum_{k=1}^{n} \mu(A_k).$$

  for any $A_1, \ldots, A_n \in \mathcal{F}$.
- $\mu : \mathcal{F} \to \mathbb{R}_+$ is *σ-sub-additive* if the same holds for $n = \infty$.
- μ is *monotone* if $(A \subseteq B \Rightarrow \mu(A) \leq \mu(B))$
- A σ-subadditive, monotone $\mu^* : 2^\Omega \to \mathbb{R}_+$ with $\mu^*(\emptyset) = 0$ is an *outer measure*.
- A set $A \subseteq \Omega$ is called *μ^*-measurable* if

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c), \qquad E \subseteq \Omega.$$

Definition 2.1

- If there is $\Omega_1, \Omega_2, \ldots \in \mathcal{F}$ with $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ and $\mu(\Omega_n) < \infty$ for all $n = 1, 2, \ldots$, then μ is *σ-finite*.
- $\mathcal{F}$ $\cap$-stable. μ is inner $\mathcal{K}-$regular if for all $A \in \mathcal{F}$

$$\mu(A) = \sup_{\mathcal{K} \ni K \subseteq A} \mu(K).$$

- $(\Omega, \mathcal{O})$ topological space, μ measure on $\mathcal{B}(\mathcal{O})$. The smallest closed set F with $\mu(F^c) = 0$ is called the *support of* μ.

Examples

- Let $\mathcal{H} = \{(a, b] : a, b \in \mathbb{Q}, a \leq b\}$. Then, $\mu((a, b]) := b - a$ defines an additive, σ-finite set function.
- Let $\omega' \in \Omega$. Then, $\delta_{\omega'}(A) := 1_{\{\omega' \in A\}}$ is a probability measure.
- $\mu := \sum_{i \in I} \delta_{\omega_i}$ is a *counting measure*.
- $\mu_i, i \in I$ measures and $a_i \in \mathbb{R}_+, i \in I$. Then, $\sum_{i \in I} a_i \mu_i$ is also a measure, e.g. the Poisson distribution on $2^{\mathbb{N}_0}$,

$$\mu_{\mathrm{Poi}(\gamma)} := \sum_{k=0}^{\infty} e^{-\gamma} \frac{\gamma^k}{k!} \cdot \delta_k,$$

the geometric distribution

$$\mu_{\mathrm{geo}(p)} := \sum_{k=1}^{\infty} (1-p)^{k-1} p \cdot \delta_k,$$

the binomial distribution

$$\mu_{B(n,p)} := \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \delta_k.$$

Unions and disjoint unions

- Lemma 2.4: $\mathcal{H}$ semi-ring, $A, A_1, \ldots, A_n \in \mathcal{H}$. Then, there are $B_1, \ldots, B_m \in \mathcal{H}$ pairwise disjoint and $A \setminus \bigcup_{i=1}^n A_i = \biguplus_{j=1}^m B_j$.
- Proof: Induction on n. If $n = 1$, clear. Assume the assertion holds for some n, i.e. there is $B_1, \ldots, B_m$ with $A \setminus \bigcup_{i=1}^n A_i = \biguplus_{j=1}^m B_j$. Then, write $B_j \setminus A_{n+1} = \biguplus_{k=1}^{k_j} C_k^j$ for $C_1^j, \ldots, C_{k_j}^j \in \mathcal{H}$. Then,

$$A \setminus \bigcup_{i=1}^{n+1} A_i = \Big(A \setminus \bigcup_{i=1}^{n} A_i\Big) \setminus A_{n+1} = \biguplus_{j=1}^{m} B_j \setminus A_{n+1} = \biguplus_{j=1}^{m} \biguplus_{k=1}^{k_j} C_k^j.$$

Set-functions on semi-rings

- Lemma 2.5: $\mathcal{H}$ semi-ring, $\mu : \mathcal{H} \to [0, \infty]$ additive. Then, m is monotone and sub-additive.
- Proof: Monotonicity for $A, B \in \mathcal{H}$ with $A \subseteq B$ and $C_1, ..., C_k \in \mathcal{H}$ with $B \setminus A = \biguplus_{i=1}^k C_i$. Write $\mu(A) \leq \mu(A) + \sum_{i=1}^k \mu(C_i) = \mu(B)$.
Claim: $\biguplus_{I \in \mathcal{I}} A_i \subseteq A \Rightarrow \sum_{i=1}^n \mu(A_i) \leq m(A)$.
Write $A \setminus \biguplus_{i=1}^n A_i = \biguplus_{j=1}^m B_j$. Then,

$$\mu(A) = \mu\Big(\biguplus_{i=1}^n A_i \uplus \biguplus_{j=1}^m B_j \Big) = \sum_{i=1}^n \mu(A_i) + \sum_{j=1}^m \mu(B_j) \geq \sum_{i=1}^n \mu(A_i).$$

Sub-additivity: To show $\mu\Big(\bigcup_{i=1}^n A_i \Big) \leq \sum_{i=1}^n \mu(A_i)$. Write

$$\mu\Big(\bigcup_{i=1}^n A_i \Big) = \mu\Big(\biguplus_{i=1}^n \Big(A_i \setminus \bigcup_{j=1}^{i-1} A_j \Big) \Big) = \sum_{k=1}^n \sum_{k=1}^{k_i} \mu(C_k^i) \leq \sum_{i=1}^n \mu(A_i).$$

Set-functions on semi-rings

- Lemma 2.5: μ is σ-additive iff μ is σ-sub-additive.
- Proof: '$\Rightarrow$': Copy the proof of sub-additivity using $n = \infty$.
  '$\Leftarrow$': Let $A = \biguplus_{i=1}^{\infty} A_i \in \mathcal{H}$.
  Then, $\sum_{i=1}^{n} \mu(A_i) \leq \mu(A)$ by monotonicity and

  $$\sum_{i=1}^{\infty} \mu(A_i) = \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} \mu(A_i) \leq \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

  by σ-sub-additivity.

Extension of set-functions on semi-rings

- Lemma 2.6: $\mathcal{H}$ semi-ring, $\mathcal{R}$ ring generated by $\mathcal{H}$, μ additive on $\mathcal{H}$. Then,
$$\widetilde{\mu}\Big(\biguplus_{i=1}^{n} A_i\Big) := \sum_{i=1}^{n} \mu(A_i)$$
$\widetilde{\mu}$ is the only additive extension of μ on $\mathcal{R}$ that coincides with μ on $\mathcal{H}$.
- Proof: Suffices to show that $\widetilde{\mu}$ is well-defined. Let $\biguplus_{i=1}^{m} A_i = \biguplus_{j=1}^{n} B_j$. Since
$$A_i = \biguplus_{j=1}^{n} A_i \cap B_j, \qquad B_j = \biguplus_{i=1}^{m} A_i \cap B_j,$$
$$\sum_{i=1}^{m} \mu(A_i) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mu(A_i \cap B_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} \mu(A_i \cap B_j) = \sum_{j=1}^{n} \mu(B_j).$$

Inclusion exclusion principle

- Proposition 2.7: μ be additive set function on ring $\mathcal{R}$ and I finite. Then for $A_i \in \mathcal{R}$, $i \in I$, it holds that

$$\mu\Big(\bigcup_{i\in I} A_i\Big) = \sum_{J\subseteq I} (-1)^{|J|+1} \mu\Big(\bigcap_{j\in J} A_j\Big)$$

  In particular, if $I = \{1, 2\}$,

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2).$$

- Proof for $|I| = 2$: $A_1 \cup A_2 = A_1 \uplus (A_2 \setminus A_1)$ and $(A_2 \setminus A_1) \uplus (A_1 \cap A_2) = A_2$.