

The background of the slide is a solid blue color with a large, faint watermark of the University of Vienna seal. The seal features a central figure, likely a scholar or saint, seated and holding a book. Above the figure are three smaller figures in niches. The seal is surrounded by Latin text in a circular border.

Measure Theory for Probabilists

2. Semi-rings, rings and σ -fields

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Definition of some set-systems

▶ $\mathcal{C} \subseteq 2^\Omega$

\mathcal{C} σ -field \implies \mathcal{C} ring \implies \mathcal{C} semi-ring.

▶ Definition 1.1: Ω set, $\emptyset \neq \mathcal{H}, \mathcal{R}, \mathcal{F} \subseteq 2^\Omega$.

▶ \mathcal{H} \cap -stable, if $(A, B \in \mathcal{H} \implies A \cap B \in \mathcal{H})$.

▶ \mathcal{H} σ - \cap -stable, if $(A_1, A_2, \dots \in \mathcal{H} \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{H})$.

▶ \mathcal{H} \cup -stable, if $(A, B \in \mathcal{H} \implies A \cup B \in \mathcal{H})$.

▶ \mathcal{H} σ - \cup -stable, if $(A_1, A_2, \dots \in \mathcal{H} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{H})$.

▶ \mathcal{H} complement-stable, if $A \in \mathcal{H} \implies A^c \in \mathcal{H}$.

▶ \mathcal{H} set-difference-stable, if $(A, B \in \mathcal{H} \implies B \setminus A \in \mathcal{H})$.

Definition of some set-systems

- ▶ We write $A \uplus B$ for $A \cup B$ if $A \cap B = \emptyset$.
- ▶ Definition 1.1: Ω set, $\emptyset \neq \mathcal{H}, \mathcal{R}, \mathcal{F} \subseteq 2^\Omega$.
 - ▶ \mathcal{H} is a *semi-ring*, if it is (i) \cap -stable and (ii) $\forall A, B \in \mathcal{H} \exists C_1, \dots, C_n \in \mathcal{H}$ with $B \setminus A = \biguplus_{i=1}^n C_i$.
 - ▶ \mathcal{R} is a *ring*, if it is \cup -stable and set-difference-stable.
 - ▶ \mathcal{F} is a σ -*field*, if $\Omega \in \mathcal{F}$, it is complement-stable and σ - \cup -stable. Then, (Ω, \mathcal{F}) is called *measurable space*.

Connections between set-systems

	\mathcal{C} semi-ring	\mathcal{C} ring	\mathcal{C} σ -field
\mathcal{C} is \cap -stable	•	○	○
\mathcal{C} is σ - \cap -stable			○
\mathcal{C} is \cup -stable		•	○
\mathcal{C} is σ - \cup -stable			•
\mathcal{C} is set-difference-stable		•	○
\mathcal{C} is complement-stable			•
$B \setminus A = \biguplus_{i=1}^n C_i$	•	○	○
$\Omega \in \mathcal{C}$			•

Examples

- ▶ Semi-ring: Let $\Omega = \mathbb{R}$. Then,

$\mathcal{H} := \{(a, b] : a, b \in \mathbb{Q}, a \leq b\}$ is a semi-ring.

- ▶ σ -algebras: Trivial examples are $\{\emptyset, \Omega\}$ and 2^Ω .
If \mathcal{F}' is a σ -field on Ω' , and $f : \Omega \rightarrow \Omega'$. Then,

$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\}$ is a σ -field on Ω .

Indeed: If $A', A'_1, A'_2, \dots \in \sigma(f)$, then
 $(f^{-1}(A'))^c = f^{-1}((A')^c) \in \sigma(f)$ and
 $\bigcup_{n=1}^{\infty} f^{-1}(A'_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} A'_n\right) \in \sigma(f)$.