Measure Theory for Probabilists
2. Semi-rings, rings and σ-fields

Peter Pfaffelhuber

January 1, 2024

Definition of some set-systems

- $\mathcal{C} \subseteq 2^\Omega$

$$\mathcal{C}\ \sigma\text{-field} \quad \Longrightarrow \quad \mathcal{C}\ \text{ring} \quad \Longrightarrow \quad \mathcal{C}\ \text{semi-ring}.$$

- Definition 1.1: Ω set, $\emptyset \neq \mathcal{H}, \mathcal{R}, \mathcal{F} \subseteq 2^\Omega$.
  - $\mathcal{H}$ $\cap$-stable, if $(A, B \in \mathcal{H} \Rightarrow A \cap B \in \mathcal{H})$.
  - $\mathcal{H}$ $\sigma - \cap$-stable, if $(A_1, A_2, ... \in \mathcal{H} \Rightarrow \bigcap_{i=n}^\infty A_n \in \mathcal{H})$.
  - $\mathcal{H}$ $\cup$-stable, if $(A, B \in \mathcal{H} \Rightarrow A \cup B \in \mathcal{H})$.
  - $\mathcal{H}$ $\sigma - \cup$-stable, if $(A_1, A_2, ... \in \mathcal{H} \Rightarrow \bigcup_{i=n}^\infty A_n \in \mathcal{H})$.
  - $\mathcal{H}$ complement-stable, if $A \in \mathcal{H} \Rightarrow A^c \in \mathcal{H}$.
  - $\mathcal{H}$ set-difference-stable, if $(A, B \in \mathcal{H} \Rightarrow B \setminus A \in \mathcal{H})$.

Definition of some set-systems

- We write $A \uplus B$ for $A \cup B$ if $A \cap B = \emptyset$.
- Definition 1.1: Ω set, $\emptyset \neq \mathcal{H}, \mathcal{R}, \mathcal{F} \subseteq 2^{\Omega}$.
  - $\mathcal{H}$ is a *semi-ring*, if it is (i) $\cap$-stable and (ii) $\forall A, B \in \mathcal{H} \exists C_1, \ldots, C_n \in \mathcal{H}$ with $B \setminus A = \biguplus_{i=1}^{n} C_i$.
  - $\mathcal{R}$ is a *ring*, if it is $\cup$-stable and set-difference-stable.
  - $\mathcal{F}$ is a σ*-field*, if $\Omega \in \mathcal{F}$, it is complement-stable and σ-$\cup$-stable. Then, $(\Omega, \mathcal{F})$ is called *measurable space*.

Connections between set-systems

	$\mathcal{C}$ semi-ring	$\mathcal{C}$ ring	$\mathcal{C}$ σ-field
$\mathcal{C}$ is $\cap$-stable	$\bullet$	$\circ$	$\circ$
$\mathcal{C}$ is σ-$\cap$-stable			$\circ$
$\mathcal{C}$ is $\cup$-stable		$\bullet$	$\circ$
$\mathcal{C}$ is σ-$\cup$-stable			$\bullet$
$\mathcal{C}$ is set-difference-stable		$\bullet$	$\circ$
$\mathcal{C}$ is complement-stable			$\bullet$
$B \setminus A = \biguplus_{i=1}^{n} C_i$	$\bullet$	$\circ$	$\circ$
$\Omega \in \mathcal{C}$			$\bullet$

Examples

- Semi-ring: Let $\Omega = \mathbb{R}$. Then,

$$\mathcal{H} := \{(a, b] : a, b \in \mathbb{Q}, a \leq b\} \text{ is a semi-ring.}$$

- σ-algebras: Trivial examples are $\{\emptyset, \Omega\}$ and 2^Ω.
If $\mathcal{F}'$ is a σ-field on Ω', and $f : \Omega \to \Omega'$. Then,

$$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\} \text{ is a } \sigma\text{-field on } \Omega.$$

Indeed: If $A', A'_1, A'_2, \ldots \in \sigma(f)$, then
$(f^{-1}(A'))^c = f^{-1}((A')^c) \in \sigma(f)$ and
$\bigcup_{n=1}^{\infty} f^{-1}(A'_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} A'_n\right) \in \sigma(f)$.