Probability Theory
20. Regular version of conditional distribution

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Problem statement

- Target: $\mathcal{G} \subseteq \mathcal{F}$; Define $\mathrm{P}(.|\mathcal{G})(\omega)$ for P-almost all $\omega \in \Omega$.
- Problem: For $A_1, A_2, \cdots \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$ and $B \in \mathcal{G}$

$$\mathrm{E}\Big[\mathrm{P}\Big(\bigcup_{n=1}^{\infty} A_n|\mathcal{G}\Big); B\Big] = \mathrm{E}\big[\mathrm{E}[1_{\bigcup_{n=1}^{\infty} A_n}|\mathcal{G}]; B\big] = \mathrm{E}[1_{\bigcup_{n=1}^{\infty} A_n}; B]$$

$$= \mathrm{E}\Big[\sum_{n=1}^{\infty} 1_{A_n}; B\Big] = \sum_{n=1}^{\infty} \mathrm{E}[1_{A_n}; B] = \sum_{n=1}^{\infty} \mathrm{E}\big[\mathrm{P}(A_n|\mathcal{G}); B\big]$$

$$= \mathrm{E}\big[\sum_{n=1}^{\infty} \mathrm{P}(A_n|\mathcal{G}); B\big]$$

so

$$\mathrm{P}\Big(\bigcup_{n=1}^{\infty} A_n|\mathcal{G}\Big) = \sum_{n=1}^{\infty} \mathrm{P}(A_n|\mathcal{G})$$

P-almost everywhere. The exception null set depends depends on the sequence $A_1, A_2, \ldots$, but there are uncountably many such sequences.

Regular version of the conditional distribution

- Let $(\Omega', \mathcal{F}')$ be a measurable space. $\kappa : \Omega \times \mathcal{F}' \to [0, 1]$ is a *stochastic kernel (from $\mathcal{F}$) to* $(\Omega', \mathcal{F}')$ if
  1. $\kappa(\omega, .)$ for all ω is a probability measure on $\mathcal{F}'$;
  2. $\kappa(., A')$ for all $A' \in \mathcal{F}'$ is $\mathcal{F}$-measurable.
- Definition 11.20: Y rv, $\mathcal{G} \subseteq \mathcal{F}$ a σ-algebra. A stochastic kernel $\kappa_{Y,\mathcal{G}}$ from $\mathcal{G}$ to $(\Omega', \mathcal{F}')$ is called *regular version of the conditional distribution* of Y, given $\mathcal{G}$, if

$$\kappa_{Y,\mathcal{G}}(\omega, B) = \mathrm{P}(Y \in B|\mathcal{G})(\omega)$$

  for P-almost all ω and every $B \in \mathcal{F}'$.

Existence of the regular version of the conditional distribution

- Theorem 11.22: (E, r) complete, separable metric space, Y an E-valued rvs.

  For $\mathcal{G} \subseteq \mathcal{F}$ there exists a regular version of the conditional distribution of Y given $\mathcal{G}$.