

The background of the slide is a solid blue color with a large, faint watermark of the University of Vienna seal. The seal features a central figure, likely a scholar or saint, seated and holding a book. Above the figure are three smaller figures in niches. The entire scene is enclosed in a circular border with Latin text. The text on the left and right sides of the seal is partially visible and reads "SIGILLUM UNIVERSITATIS VIENNAE".

# Probability Theory

## 20. Regular version of conditional distribution

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June 25, 2024

## Problem statement

- ▶ Target:  $\mathcal{G} \subseteq \mathcal{F}$ ; Define  $P(\cdot|\mathcal{G})(\omega)$  for P-almost all  $\omega \in \Omega$ .
- ▶ Problem: For  $A_1, A_2, \dots \in \mathcal{F}$  with  $A_i \cap A_j = \emptyset$  and  $B \in \mathcal{G}$

$$\begin{aligned} E\left[P\left(\bigcup_{n=1}^{\infty} A_n|\mathcal{G}\right); B\right] &= E\left[E[1_{\bigcup_{n=1}^{\infty} A_n}|\mathcal{G}]; B\right] = E[1_{\bigcup_{n=1}^{\infty} A_n}; B] \\ &= E\left[\sum_{n=1}^{\infty} 1_{A_n}; B\right] = \sum_{n=1}^{\infty} E[1_{A_n}; B] = \sum_{n=1}^{\infty} E[P(A_n|\mathcal{G}); B] \\ &= E\left[\sum_{n=1}^{\infty} P(A_n|\mathcal{G}); B\right] \end{aligned}$$

so 
$$P\left(\bigcup_{n=1}^{\infty} A_n|\mathcal{G}\right) = \sum_{n=1}^{\infty} P(A_n|\mathcal{G})$$

P-almost everywhere. The exception null set depends on the sequence  $A_1, A_2, \dots$ , but there are uncountably many such sequences.

## Regular version of the conditional distribution

- ▶ Let  $(\Omega', \mathcal{F}')$  be a measurable space.  $\kappa : \Omega \times \mathcal{F}' \rightarrow [0, 1]$  is a *stochastic kernel (from  $\mathcal{F}$ ) to  $(\Omega', \mathcal{F}')$*  if
  1.  $\kappa(\omega, \cdot)$  for all  $\omega$  is a probability measure on  $\mathcal{F}'$ ;
  2.  $\kappa(\cdot, A')$  for all  $A' \in \mathcal{F}'$  is  $\mathcal{F}$ -measurable.
- ▶ Definition 11.20:  $Y$  rv,  $\mathcal{G} \subseteq \mathcal{F}$  a  $\sigma$ -algebra. A stochastic kernel  $\kappa_{Y, \mathcal{G}}$  from  $\mathcal{G}$  to  $(\Omega', \mathcal{F}')$  is called *regular version of the conditional distribution* of  $Y$ , given  $\mathcal{G}$ , if

$$\kappa_{Y, \mathcal{G}}(\omega, B) = P(Y \in B | \mathcal{G})(\omega)$$

for P-almost all  $\omega$  and every  $B \in \mathcal{F}'$ .

# Existence of the regular version of the conditional distribution

- ▶ Theorem 11.22:  $(E, r)$  complete, separable metric space,  $Y$  an  $E$ -valued rvs.

For  $\mathcal{G} \subseteq \mathcal{F}$  there exists a regular version of the conditional distribution of  $Y$  given  $\mathcal{G}$ .