Probability Theory
18. The case $\mathcal{G} = \sigma(X)$ and some examples

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Repetition

- Let X, Y be rvs with $Y \in \mathcal{L}^1$. Then there is an as unique, X-measurable rv $\mathrm{E}[Y|X] := \mathrm{E}[Y|\sigma(X)]$ such that
$$\mathrm{E}[Y, X \in A] = \mathrm{E}[\mathrm{E}[Y|X], X \in A].$$
- Proposition 11.7: If there is φ measurable such that
$$\mathrm{E}[Y, X \in A] = \mathrm{E}[\varphi(X), X \in A].$$
then $\varphi(X) = \mathrm{E}[Y|X]$ as.
- Lemma 7.2: Let X be rv and Z a R-valued rv. Then, Z is $\sigma(X)$-measurable iff φ is measurable with $\varphi \circ X = Z$.

Example: random success probability

- Example: $U \sim U([0,1])$; given U let $X \sim B(n, U)$. Then

$$\mathrm{P}(X = k | U) = \binom{n}{k} U^k (1-U)^{n-k}$$

since

$$\mathrm{E}[1_{X=k}; U \in I] = \mathrm{P}(X = k, U \in I) = \int_I \binom{n}{k} u^k (1-u)^{n-k} du$$
$$= \mathrm{E}\Big[\binom{n}{k} U^k (1-U)^{n-k}; U \in I\Big].$$

Example: sums of independent rvs

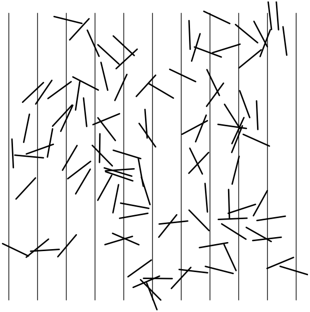
- $X_1, X_2, \ldots$ uiv, $\mu = \mathrm{E}[X_1]$ and $S_n := X_1 + \cdots + X_n$. Then

$$\mathrm{E}[S_n|X_1] = \mathrm{E}[X_1|X_1] + \mathrm{E}[X_2 + \cdots + X_n|X_1] = X_1 + (n-1)\mu,$$

$$\mathrm{E}[X_1|S_n] = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[X_i|S_n] = \frac{1}{n}\mathrm{E}[S_n|S_n] = \frac{1}{n}S_n.$$

Buffon's needle problem

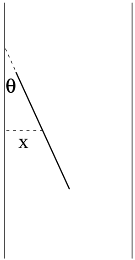
$$Z := \begin{cases} 1, & \text{if the needle intersects a straight line} \\ 0, & \text{else} \end{cases}.$$



Goal: Calculate $\mathrm{P}(Z = 1)$!

Buffon's needle problem

$$Z := \begin{cases} 1, & \text{if the needle intersects a straight line} \\ 0, & \text{else} \end{cases}.$$



$$\begin{aligned} & P(Z = 1|\Theta) \\ & \quad = P(X \leq \tfrac{1}{2}\sin(\Theta) \\ & \qquad \text{or } X \geq 1 - \tfrac{1}{2}\sin(\Theta)|\Theta) \\ & \quad = \sin(\Theta), \\ P(Z = 1) & = E[P(Z = 1|\Theta)] = E[\sin(\Theta)] \\ & = \frac{2}{\pi}\int_0^{\pi/2} (\sin(\theta) d\theta = \frac{2}{\pi}. \end{aligned}$$

Search in lists

- Given: n names of persons from r cities.

  Store the names in r unordered lists;

  Each person comes with from city j with prob p_j;

  Wanted: Entry of a randomly drawn person;

  If person not yet recorded, you need X comparisons.

  J : number of the city from which the person to be

  : searched comes;

  Z_j : number of people from city j in the lists;

  $L = Z_J$: number of unsuccessful comparisons;

$$\mathrm{E}[L] = \mathrm{E}[\mathrm{E}[Z_J | J]] = n\mathrm{E}[p_J] = n \cdot \sum_{j=1}^{r} p_j^2 .$$