Probability Theory
16. Multidimensional weak limits

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Multidimensional normal distribution

- Definition 10.14: $\mu \in \mathbb{R}^d$, $C \in \mathbb{R}^{d \times d}$ symmetric, strictly positive definite. The *d-dimensional normal distribution* $N_{\mu,C}$ on $\mathbb{R}^d$ has the density

$$f_{\mu,C}(x) = \frac{1}{\sqrt{(2\pi)^d \det(C)}} \exp\Big(-\tfrac{1}{2}(x-\mu)C^{-1}(x-\mu)^\top \Big).$$

- As is well known

$$\mathrm{E}[tX] = \sum_{i=1}^{d} t_i \mathrm{E}[X_j] = t\mu^\top,$$

$$\mathrm{V}[tX] = \sum_{i,j=1}^{d} t_i t_j \mathrm{COV}[X_i, X_j] = tCt^\top > 0,$$

$$\mathrm{E}[e^{itX}] = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}.$$

Properties

- Proposition 10.15: The following are equivalent:
  1. $X \sim N_{\mu,C}$;
  2. $tX^\top \sim N_{t\mu^\top, tCt^\top}$ for each $t \in \mathbb{R}^d$;
  3. $\psi_X(t) = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}$ for each $t \in \mathbb{R}^d$.

  In each of these cases, for $C = AA^\top$
  4. $X \overset{d}{=} AY + \mu$ for $Y \sim N_{0,I}$
  5. $\mathrm{E}[X_i] = \mu_i$ for $i = 1, \dots, d$
  6. $\mathrm{COV}[X_i, X_j] = C_{ij}$ for $i, j = 1, \dots, d$

  4. for $t \in \mathbb{R}^d$

$$\mathrm{E}[e^{it(AY+\mu)^\top}] = e^{it\mu^\top} e^{-\frac{1}{2}tAA^\top t^\top} = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}$$

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  1. $X \sim N_{\mu,C}$;
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  1.$\Leftrightarrow$ 3.: Clear, since characteristic functions are

  distribution-determining.

  2.$\Leftrightarrow$ 3.: The equation

$$\mathrm{E}[e^{itX}] = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}$$

  can be used in both directions.

Cramér-Wold device

- Proposition 10.17: Let $X, X_1, X_2, \dots$ be rvs with values in $\mathbb{R}^d$. Then,

$$X_n \xRightarrow{n\to\infty} X \qquad \iff \qquad tX_n \xRightarrow{n\to\infty} tX, \quad t \in \mathbb{R}^d.$$

'$\Rightarrow$': Clear, because for $t \in \mathbb{R}^d$ the map $x \mapsto f(tx)$ is continuous.

'$\Leftarrow$': Let π_i be the projection onto the i-th coordinate. Since $(\pi_i X_n)_{n=1,2,\dots}$ is tight for all i, $(X_n)_{n=1,2,\dots}$ is tight. Further $\mathrm{E}[e^{itX_n}] \xrightarrow{n\to\infty} \mathrm{E}[e^{itX}]$ for all $t \in \mathbb{R}^d$ and Proposition 10.27.

Multidimensional CLT

- Theorem 10.18: $X_1, X_2, \ldots$ iid, values in $\mathbb{R}^d$ with $E[X_n] = \mu \in \mathbb{R}^d$ and $COV[X_{n,i}, X_{n,j}] = C_{ij}$ for $i, j = 1, \ldots, d$ and $S_n = \sum_{i=1}^n X_i$. Then applies

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{n\to\infty} X \sim N_{0,C}.$$

For $t \in \mathbb{R}^d$ application of the ZGS to $tX_1, tX_2, \ldots$ to. Thus

$$t\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{n\to\infty} tX.$$

Since t was arbitrary, the statement follows.