

The background of the slide features a large, light blue watermark of the University of Bonn seal. The seal is circular and contains a central figure of a seated woman, likely the personification of Wisdom, holding a book. Above her are three portraits of men. The seal is surrounded by Latin text: 'SIGILLUM UNIVERSITATIS BONNENSIS' at the top and 'MDCCCXXXIII' at the bottom.

# Probability Theory

## 16. Multidimensional weak limits

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# Multidimensional normal distribution

- ▶ Definition 10.14:  $\mu \in \mathbb{R}^d$ ,  $C \in \mathbb{R}^{d \times d}$  symmetric, strictly positive definite. The  $d$ -dimensional normal distribution  $N_{\mu,C}$  on  $\mathbb{R}^d$  has the density

$$f_{\mu,C}(x) = \frac{1}{\sqrt{(2\pi)^d \det(C)}} \exp\left(-\frac{1}{2}(x - \mu)C^{-1}(x - \mu)^\top\right).$$

- ▶ As is well known

$$E[tX] = \sum_{i=1}^d t_i E[X_i] = t\mu^\top,$$

$$V[tX] = \sum_{i,j=1}^d t_i t_j \text{COV}[X_i, X_j] = tCt^\top > 0,$$

$$E[e^{itX}] = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}.$$

# Properties

► Proposition 10.15: The following are equivalent:

1.  $X \sim N_{\mu, C}$ ;
2.  $tX^\top \sim N_{t\mu^\top, tCt^\top}$  for each  $t \in \mathbb{R}^d$ ;
3.  $\psi_X(t) = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}$  for each  $t \in \mathbb{R}^d$ .

In each of these cases, for  $C = AA^\top$

4.  $X \stackrel{d}{=} AY + \mu$  for  $Y \sim N_{0, I}$
  5.  $E[X_i] = \mu_i$  for  $i = 1, \dots, d$
  6.  $\text{COV}[X_i, X_j] = C_{ij}$  for  $i, j = 1, \dots, d$
4. for  $t \in \mathbb{R}^d$

$$E[e^{it(AY + \mu)^\top}] = e^{it\mu^\top} e^{-\frac{1}{2}tAA^\top t^\top} = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}$$

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1.  $\Leftrightarrow$  3.: Clear, since characteristic functions are distribution-determining.

2.  $\Leftrightarrow$  3.: The equation

$$E[e^{itX}] = e^{it\mu^\top} e^{-\frac{1}{2}tCt^\top}$$

can be used in both directions.

## Cramér-Wold device

- ▶ Proposition 10.17: Let  $X, X_1, X_2, \dots$  be rvs with values in  $\mathbb{R}^d$ .

Then,

$$X_n \xrightarrow{n \rightarrow \infty} X \quad \iff \quad tX_n \xrightarrow{n \rightarrow \infty} tX, \quad t \in \mathbb{R}^d.$$

' $\Rightarrow$ ': Clear, because for  $t \in \mathbb{R}^d$  the map  $x \mapsto f(tx)$  is continuous.

' $\Leftarrow$ ': Let  $\pi_i$  be the projection onto the  $i$ -th coordinate. Since  $(\pi_i X_n)_{n=1,2,\dots}$  is tight for all  $i$ ,  $(X_n)_{n=1,2,\dots}$  is tight. Further  $E[e^{itX_n}] \xrightarrow{n \rightarrow \infty} E[e^{itX}]$  for all  $t \in \mathbb{R}^d$  and Proposition 10.27.

# Multidimensional CLT

- Theorem 10.18:  $X_1, X_2, \dots$  iid, values in  $\mathbb{R}^d$  with  $E[X_n] = \mu \in \mathbb{R}^d$  and  $\text{COV}[X_{n,i}, X_{n,j}] = C_{ij}$  for  $i, j = 1, \dots, d$  and  $S_n = \sum_{i=1}^n X_i$ . Then applies

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} X \sim N_{0,C}.$$

For  $t \in \mathbb{R}^d$  application of the ZGS to  $tX_1, tX_2, \dots$  to. Thus

$$t \frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} tX.$$

Since  $t$  was arbitrary, the statement follows.