Probability Theory
15. The Central Limit Theorem

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The Central Limit Theorem (Lindeberg-Feller)

- Theorem 10.8: $(X_{nj})_{n=1,2,\ldots,j=1,\ldots,m_n}$ rvs, such that

  $X_{n1}, \ldots, X_{nm_n}$ are independent, $n = 1, 2, \ldots$, $X \sim N(\mu, \sigma^2)$

  with

  $$\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}] \xrightarrow{n\to\infty} \mu, \qquad \sum_{j=1}^{m_n} \mathrm{V}[X_{nj}] \xrightarrow{n\to\infty} \sigma^2.$$

  Then are equivalent:

  1. $\sum_{j=1}^{m_n} X_{nj} \xRightarrow{n\to\infty} X$ and $\sup_{j=1,\ldots,m_n} \mathrm{V}[X_{nj}] \xrightarrow{n\to\infty} 0$,
  2. $\sum_{j=1}^{m_n} \mathrm{E}[(X_{nj} - \mathrm{E}[X_{nj}])^2; |X_{nj} - \mathrm{E}[X_{nj}]| > \varepsilon] \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

     (Lindeberg condition)

The Central Limit Theorem (Lindeberg-Feller)

- Corollary 10.9: $X_1, X_2, \ldots$ iid with $\mathrm{E}[X_1] = \mu, \mathrm{V}[X_1] = \sigma^2 > 0$, $S_n := \sum_{k=1}^n X_k$ and $X \sim N(0,1)$. Then,

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xRightarrow{n\to\infty} X.$$

Proof: $m_n = n$ and $X_{nj} = \frac{X_j - \mu}{\sqrt{n\sigma^2}}$ fulfills the conditions of the CLT with $\mu = 0, \sigma^2 = 1$. Furthermore

$$\sum_{j=1}^n \mathrm{E}[X_{nj}^2; |X_{nj}| > \varepsilon] = \frac{1}{\sigma^2}\mathrm{E}[(X_1 - \mu)^2; |X_1 - \mu| > \varepsilon\sqrt{n\sigma^2}] \xrightarrow{n\to\infty} 0$$

due to dominated convergence.

The Lyapunoff condition

$$\left(\exists \delta > 0 : \sum_{j=1}^{m_n} \mathsf{E}\left[|X_{nj} - \mathsf{E}[X_{nj}]|^{2+\delta}\right] \xrightarrow{n\to\infty} 0\right) \implies \text{Lindeberg}$$

Indeed: For $\varepsilon > 0$,

$$x^2 1_{|x|>\varepsilon} \le \frac{|x|^{2+\delta}}{\varepsilon^\delta} 1_{|x|>\varepsilon} \le \frac{|x|^{2+\delta}}{\varepsilon^\delta}.$$

With $\mathsf{E}[X_{nj}] = 0$

$$\sum_{j=1}^{m_n} \mathsf{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \le \frac{1}{\varepsilon^\delta} \sum_{j=1}^{m_n} \mathsf{E}[|X_{nj}|^{2+\delta}] \xrightarrow{n\to\infty} 0.$$

Preliminaries

- Lemma 10.11: For $z_1, \ldots, z_n, z_1', \ldots, z_n' \in \mathbb{C}$ with $|.| \leq 1$

$$\left| \prod_{k=1}^{n} z_k - \prod_{k=1}^{n} z_k' \right| \leq \sum_{k=1}^{n} |z_k - z_k'|.$$

- Lemma 10.12: For $t \in \mathbb{C}$ and $n \in \mathbb{Z}_+$,

$$\left| e^{it} - \sum_{k=0}^{n} \frac{(it)^k}{k!} \right| \leq \frac{2|t|^n}{n!} \wedge \frac{|t|^{n+1}}{(n+1)!}.$$

- We write

$$a \lesssim b \qquad \iff \qquad \exists C > 0 : a \leq Cb.$$

- Wlog $\mathrm{E}[X_{nj}] = 0$, $\sigma_{nj} = \mathrm{V}[X_{nj}], \sigma^2 = 1$,

$$\psi_{nj}(t) := \psi_{X_{nj}}(t) = \mathrm{E}[e^{itX_{nj}}],$$

$$\widetilde{\psi}_{nj}(t) = \mathrm{E}[e^{itZ_{nj}}] \text{ for } Z_{nj} \sim N(0, \sigma_{nj}^2).$$

The Central Limit Theorem (Lindeberg-Feller)

- Theorem 10.8:

$$\sum_{j=1}^{m_n} \mathsf{E}[X_{nj}] \xrightarrow{n\to\infty} 0, \qquad \sum_{j=1}^{m_n} \sigma_{nj}^2 \xrightarrow{n\to\infty} 1.$$

Then are equivalent:

1. $\sum_{j=1}^{m_n} X_{nj} \xRightarrow{n\to\infty} X$ and $\sup_{j=1,\dots,m_n} \sigma_{nj} \xrightarrow{n\to\infty} 0$,
2. $\sum_{j=1}^{m_n} \mathsf{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

2. $\Rightarrow$ 1.: For $\varepsilon > 0$,

$$\sup_{j=1,\dots,m_n} \sigma_{nj}^2 \le \varepsilon^2 + \sup_{j=1,\dots,m_n} \mathsf{E}[X_{nj}^2; |X_{nj}| > \varepsilon]$$

$$\le \varepsilon^2 + \sum_{j=1}^{m_n} \mathsf{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \xrightarrow{n\to\infty} \varepsilon^2,$$

The Central Limit Theorem (Lindeberg-Feller)

- Theorem 10.8:

$$\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}] \xrightarrow{n\to\infty} 0, \qquad \sum_{j=1}^{m_n} \sigma_{nj}^2 \xrightarrow{n\to\infty} 1.$$

Then are equivalent:

1. $\sum_{j=1}^{m_n} X_{nj} \xRightarrow{n\to\infty} X$ and $\sup_{j=1,\dots,m_n} \sigma_{nj} \xrightarrow{n\to\infty} 0$,
2. $\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

$$\begin{aligned}\Big|\prod_{j=1}^{m_n} \psi_{nj}(t) - \prod_{j=1}^{m_n} \widetilde{\psi}_{nj}(t)\Big| &\le \sum_{j=1}^{m_n} |\psi_{nj}(t) - \widetilde{\psi}_{nj}(t)| \\ &\le \sum_{j=1}^{m_n} |\psi_{nj}(t) - 1 + \tfrac{1}{2}t^2\sigma_{nj}^2| + \sum_{j=1}^{m_n} |\widetilde{\psi}_{nj}(t) - 1 + \tfrac{1}{2}t^2\sigma_{nj}^2| \\ &\lesssim 2\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2(1 \wedge |X_{nj}|]) + \sum_{j=1}^{m_n} |e^{-\frac{1}{2}\sigma_{nj}^2 t^2} - 1 + \tfrac{1}{2}t^2\sigma_{nj}^2|.\end{aligned}$$

The Central Limit Theorem (Lindeberg-Feller)

- Theorem 10.8:

$$\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}] \xrightarrow{n\to\infty} 0, \qquad \sum_{j=1}^{m_n} \sigma_{nj}^2 \xrightarrow{n\to\infty} 1.$$

  Then are equivalent:

  1. $\sum_{j=1}^{m_n} X_{nj} \xRightarrow{n\to\infty} X$ and $\sup_{j=1,\dots,m_n} \sigma_{nj} \xrightarrow{n\to\infty} 0$,
  2. $\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

$$\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2(1 \wedge |X_{nj}|)] \leq \varepsilon \sum_{j=1}^{m_n} \sigma_{nj}^2 + \sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \xrightarrow{n\to\infty} \varepsilon$$

$$\sum_{j=1}^{m_n} |e^{-\frac{1}{2}\sigma_{nj}^2 t^2} - 1 + \tfrac{1}{2}t^2\sigma_{nj}^2| \lesssim \sum_{j=1}^{m_n} \sigma_{nj}^4 \leq \sigma_n^2 \sup_{j=1,\dots,m_n} \sigma_{nj}^2 \xrightarrow{n\to\infty} 0$$

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Then are equivalent:

1. $\sum_{j=1}^{m_n} X_{nj} \overset{n\to\infty}{\Longrightarrow} X$ and $\sup_{j=1,\dots,m_n} \sigma_{nj} \xrightarrow{n\to\infty} 0$,
2. $\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

1. $\Rightarrow$ 2.:

$$\sum_{j=1}^{m_n} \mathrm{E}[\cos(tX_{nj}) - 1] = \mathrm{Re} \sum_{j=1}^{m_n} (\psi_{nj}(t) - 1)$$

$$\approx \mathrm{Re} \sum_{j=1}^{m_n} \log \psi_{nj}(t) \xrightarrow{n\to\infty} -\frac{t^2}{2}$$

The Lindeberg-Feller theorem

- Theorem 10.8:

$$\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}] \xrightarrow{n\to\infty} 0, \qquad \sum_{j=1}^{m_n} \sigma_{nj}^2 \xrightarrow{n\to\infty} 1.$$

  Then are equivalent:

  1. $\sum_{j=1}^{m_n} X_{nj} \xRightarrow{n\to\infty} X$ and $\sup_{j=1,\dots,m_n} \sigma_{nj} \xrightarrow{n\to\infty} 0$,
  2. $\sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

  Because of $\cos(\theta) \approx 1 - \frac{\theta^2}{2}$

$$\limsup_{n\to\infty} \sum_{j=1}^{m_n} \mathrm{E}[X_{nj}^2; |X_{nj}| > \varepsilon] \approx \limsup_{n\to\infty} \frac{2}{t^2} \sum_{j=1}^{m_n} \mathrm{E}[1 - \cos(tX_{nj}); |X_{nj}| > \varepsilon]$$

$$\leq \limsup_{n\to\infty} \frac{2}{t^2} \sum_{j=1}^{m_n} \mathrm{P}[|X_{nj}| > \varepsilon] \leq \frac{2}{\varepsilon^2 t^2} \limsup_{n\to\infty} \sum_{j=1}^{m_n} \sigma_{nj} = \frac{2}{\varepsilon^2 t^2}.$$