Probability Theory
14. Poisson convergence

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Example: Binomial $\Rightarrow$ Poisson

- Let $p_1, p_2, \ldots \in [0,1]$ such that $np_n \to \lambda \geq 0$.

  Then, $B(n, p_n) \xrightarrow{n\to\infty} \text{Poi}(\lambda)$.

  Indeed:

$$\begin{aligned}
\psi_{B(n,p_n)}(t) &= \Big(1 - p_n\big(1 - e^{it}\big)\Big)^n \\
&= \Big(1 - \frac{n \cdot p_n}{n}\big(1 - e^{it}\big)\Big)^n \\
&\xrightarrow{n\to\infty} \exp\big(-\lambda(1 - e^{it})\big) = \psi_{\text{Poi}(\lambda)}(t).
\end{aligned}$$

The generating function

- Let X be rv with values in $\mathbb{Z}_+$. Then,

$$z \mapsto \varphi_X(z) := \mathsf{P}[z^X] = \sum_{k=0}^{\infty} z^k \mathsf{P}(X = k)$$

is called generation function (of the distribution) of X. With $z = e^{-t}$,

$$\mathcal{L}_X(t) = \mathsf{P}[e^{-tX}] = \mathsf{P}[z^X] = \varphi_X(z).$$

- Generating function is distribution-determining;

  Weak convergence $\iff$ Convergence of the gener. fcts.;
- Note that

$$\varphi_X'(1) = \sum_{k=0}^{\infty} k z^{k-1} \mathsf{P}(X = k)\Big|_{z=1} = \sum_{k=0}^{\infty} k \mathsf{P}(X = k) = \mathsf{P}[X].$$

Asymptotic negligibility

- Definition 10.4: A family $(X_{nj})_{n=1,2,\dots,n,j=1,\dots,m_n}$ with $m_1, m_2, \dots \in \mathbb{N}$ is *asymptotically negligible* if $X_{n1}, \dots, X_{n,m_n}$ is independent, $n = 1, 2, \dots$ and for all $\varepsilon > 0$

$$\sup_{j=1,\dots,m_n} \mathrm{P}(|X_{nj}| > \varepsilon) \xrightarrow{n\to\infty} 0.$$

  If $X_{ij} \geq 0$ for all i, j, then $m_n = \infty$ is also allowed.

- For $\mathbb{Z}$-valued rvs this is equivalent to

$$\left.\begin{array}{c} \inf_{j=1,\dots,m_n} \mathrm{P}(|X_{nj}| = 0) \\ \inf_{j=1,\dots,m_n} \mathrm{E}[|X_{nj}| \wedge 1] \\ \inf_{j=1,\dots,m_n} \varphi_{X_{n_j}}(0) \end{array}\right\} \xrightarrow{n\to\infty} 1.$$

A lemma

- Lemma 10.6: $(\lambda_{nj})_{n=1,2,\dots,j=1,\dots,m_n}$ non-negative, $\lambda \geq 0$. Then

$$\prod_{j=1}^{m_n}(1-\lambda_{nj}) \xrightarrow{n\to\infty} e^{-\lambda} \qquad \iff \qquad \sum_{j=1}^{m_n}\lambda_{nj} \xrightarrow{n\to\infty} \lambda.$$

Proof: $\log(1-x) = -x + o(x)$. LHS equivalent to

$$-\lambda = \lim_{n\to\infty}\sum_{j=1}^{m_n}\log(1-\lambda_{nj}) = -\lim_{n\to\infty}\sum_{j=1}^{m_n}\lambda_{nj}\left(1-\frac{\varepsilon(\lambda_{nj})}{\lambda_{nj}}\right)$$

$$= -\lim_{n\to\infty}\sum_{j=1}^{m_n}\lambda_{nj}.$$

Poisson convergence

- Theorem 10.5: $(X_{nj})_{n=1,2,\ldots,n,j=1,\ldots,m_n}$ asymptotically negligible, $\mathbb{Z}_+$-valued, $X \sim \text{Poi}(\lambda)$. Then

$$\sum_{j=1}^{m_n} X_{nj} \xRightarrow{n\to\infty} X \qquad \iff \qquad \begin{pmatrix} \sum_{j=1}^{m_n} \mathrm{P}(X_{nj} > 1) \xrightarrow{n\to\infty} 0, \\ \sum_{j=1}^{m_n} \mathrm{P}(X_{nj} = 1) \xrightarrow{n\to\infty} \lambda. \end{pmatrix}$$

$\Leftarrow$: $\varphi_{n,j} := \varphi_{X_{n,j}}$, to show $\prod_{j=1}^{m_n} \varphi_{nj}(z) \xrightarrow{n\to\infty} e^{-\lambda(1-z)}$ or

$$A_n(z) := \sum_{j=1}^{m_n} (1 - \varphi_{nj}(z)) \xrightarrow{n\to\infty} \lambda(1-z),$$

We write

$$A_n(z) = \sum_{j=1}^{m_n} 1 - \mathrm{P}(X_{nj} = 0) - z\mathrm{P}(X_{nj} = 1) + o(1)$$

$$= \sum_{j=1}^{m_n} (1-z)\mathrm{P}(X_{nj} = 1) \xrightarrow{n\to\infty} \lambda(1-z).$$

Poisson convergence of geometrically distributed rv

- $Y_{nj} + 1 \sim \text{geo}(p_n)$, $j = 1, \ldots, n$, $n = 1, 2, \ldots$ We set

  $Y_n := \sum_{j=1}^{n} Y_{nj}$,

  Number of failures before the nth success.

  If $Y \sim \text{Poi}(\lambda)$ and $(1 - p_n) \cdot n \xrightarrow{n\to\infty} \lambda$, then $Y_n \xrightarrow{n\to\infty} Y$.

  Indeed:

$$\sum_{j=1}^{n} \mathrm{P}(Y_{nj} = 1) = n(1 - p_n)p_n \xrightarrow{n\to\infty} \lambda,$$

$$\sum_{j=1}^{n} \mathrm{P}(Y_{nj} > 1) = n(1 - p_n)^2 \xrightarrow{n\to\infty} 0$$