Probability Theory
13. Lévy's Theorem

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June 5, 2024

Weak convergence and separating function classes

- Proposition 9.27: Let $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2, \dots \in \mathcal{P}(E)$ and $\mathcal{M} \subseteq \mathcal{C}_b(E)$ separating. Then the following are equivalent:
  1. $\mathbf{P}_n \xRightarrow{n\to\infty} \mathbf{P}$.
  2. $(\mathbf{P}_n)_{n=1,2,\dots}$ is tight and
  $$\mathbf{P}_n[f] \xRightarrow{n\to\infty} \mathbf{P}[f] \text{ for all } f \in \mathcal{M}.$$

  1. $\Rightarrow$ 2.: Clear; 2. $\Rightarrow$ 1.: Assume 2. but not 1. Then, there is $\varepsilon > 0$, some $f \in \mathcal{C}_b(E)$ and $(n_k)_{k=1,2,\dots}$, such that
  $$|\mathbf{P}_{n_k}[f] - \mathbf{P}[f]| > \varepsilon \text{ for all } k.$$
  There is $(n_{k_\ell})_{\ell=1,2,\dots}$ and $\mathbf{Q} \in \mathcal{P}(E)$ with $\mathbf{P}_{n_{k_\ell}} \xRightarrow{\ell\to\infty} \mathbf{Q}$.
  $$|\mathbf{P}[f] - \mathbf{Q}[f]| \geq \varepsilon, \qquad \mathbf{P}[g] = \lim_{\ell\to\infty} \mathbf{P}_{n_{k_\ell}}[g] = \mathbf{Q}[g], g \in \mathcal{M}.$$

Tightness and the characteristic function

- Lemma 9.28: $\mathbf{P} \in \mathcal{P}(\mathbb{R})$. Then for all $r > 0$

$$\mathbf{P}((-\infty; -r] \cup [r; \infty)) \leq \frac{r}{2} \int_{-2/r}^{2/r} (1 - \psi_{\mathbf{P}}(t))dt,$$

$$\mathbf{P}([-r; r]) \geq 2r \int_{-1/r}^{1/r} |\psi_{\mathbf{P}}(t)|dt.$$

$\sin(x)/x \leq 1$ for $x \leq 2$, $\sin x \leq x/2$ for $x \geq 2$. With $X \sim \mathbf{P}$

$$\begin{aligned}\int_{-c}^{c} (1 - \psi_{\mathbf{P}}(t))dt = \mathbf{P}\Big[\int_{-c}^{c} (1 - e^{itX})dt\Big] &= \mathbf{P}\Big[2c - \frac{1}{iX} e^{itX}\Big|_{t=-c}^{c}\Big] \\ &= 2c\mathbf{P}\Big[1 - \frac{\sin(cX)}{cX}\Big] \\ &\geq 2c\mathbf{P}\Big[1 - \frac{\sin(cX)}{cX}; |cX| \geq 2\Big] \\ &\geq c \cdot \mathbf{P}(|cX| \geq 2) = c\mathbf{P}((-\infty; -\tfrac{2}{c}] \cup [\tfrac{2}{c}; \infty)),\end{aligned}$$

and the assertion follows with $c = 2/r$.

Uniform continuity

- Definition 9.29: $\mathcal{M} \subset \mathcal{C}(\mathbb{R}^d)$ is called uniformly continuous if
$$\sup_{f\in\mathcal{M}} |f(y) - f(x)| \xrightarrow{y\to x} 0.$$
- Lemma 9.31: Let $f, f_1, f_2, \dots \in \mathcal{C}(\mathbb{R}^d)$ such that $f_n \xrightarrow{n\to\infty} f$.

f continuous in 0 $\iff$ $(f_n)_{n=1,2,\dots}$ uniformly continuous in 0.

Proof: $\Leftarrow$: $|f(t) - f(0)| \le \limsup_{n\to\infty} |f_n(t) - f_n(0)| \xrightarrow{t\to 0} 0.$

$$\begin{aligned}\Rightarrow: \quad & \limsup_{n\to\infty} |f_n(t) - f_n(0)| \\ & \le \limsup_{n\to\infty} |f_n(t) - f(t)| + |f(t) - f(0)| + |f(0) - f_n(0)| \\ & = |f(t) - f(0)| \xrightarrow{t\to 0} 0.\end{aligned}$$

Tightness and uniform continuity

- Proposition 9.32: $(\mathbf{P}_i)_{i\in I}$ family in $(\mathbf{R}^d)$. If $(\psi_{\mathbf{P}_i})_{i\in I}$ is is uniformly continuous in 0, then $(\mathbf{P}_i)_{i\in I}$ is tight.

  Proof for $d = 1$:

$$\sup_{i\in I} |1 - \psi_{\mathbf{P}_i}(t)| \xrightarrow{t\to 0} 0,$$

so

$$\begin{aligned}\sup_{r>0}\inf_{i\in I} \mathbf{P}_i([-r;r]) &\geq 1 - \inf_{r>0}\sup_{i\in I}\frac{r}{2}\int_{-2/r}^{2/r}(1-\psi_{\mathbf{P}_i}(t))dt \\ &\geq 1 - \inf_{r>0}\frac{r}{2}\int_{-2/r}^{2/r}\sup_{i\in I}|1-\psi_{\mathbf{P}_i}(t)|dt = 1.\end{aligned}$$

Lévy's continuity theorem

- Theorem 9.33: $\mathbf{P}_1, \mathbf{P}_2, \dots \in \mathcal{P}(\mathbb{R}^d)$ and $\psi : \mathbb{R}^d \to \mathbb{C}$, so that $\psi_{\mathbf{P}_n}(t) \xrightarrow{n\to\infty} \psi(t)$ for all $t \in \mathbb{R}^d$.
  (ψ cont in 0) $\Rightarrow$ ($\mathbf{P}_n \xRightarrow{n\to\infty} \mathbf{P}$ for a $\mathbf{P} \in \mathcal{P}(\mathbb{R}^d)$ with $\psi_{\mathbf{P}} = \psi$)
  Proof: $(\psi_{\mathbf{P}_n})_{n=1,2,\dots}$ in 0 is uniformly continuous, so that $(\mathbf{P}_n)_{n=1,2,\dots}$ is tight. Let $(n_k)_{k=1,2,\dots}$ and $\mathbf{P} \in \mathcal{P}(\mathbb{R}^d)$, so that $\mathbf{P}_{n_k} \xRightarrow{k\to\infty} \mathbf{P}$. Since . $x \mapsto e^{itx} \in \mathcal{C}_b(\mathbb{R})$, we have $\psi_{\mathbf{P}_{n_k}}(t) \xrightarrow{k\to\infty} \psi_{\mathbf{P}}(t)$ for all $t \in \mathbb{R}^d$. In particular,

$$\psi_P(t) = \lim \psi_{P_n}(t) = \psi(t),$$

Theorem of deMoivre-Laplace

$$S_n^* := \frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow{n\to\infty} Z \sim N(0,1).$$

- Since $\mathbf{E}[e^{itS_n}] = \sum_k \binom{n}{k}(pe^{it})^k q^{n-k} = (q + pe^{it})^n$

$$\begin{aligned}
\psi_{S_n^*}(t) &= \exp\Big(-it\sqrt{\frac{np}{q}}\Big) \cdot \psi_{B(n,p)}\Big(\frac{t}{\sqrt{npq}}\Big) \\
&= \exp\Big(-it\sqrt{\frac{np}{q}}\Big)\Big(q + p\exp\Big(\frac{it}{\sqrt{npq}}\Big)\Big)^n \\
&= \Big(q\exp\Big(-it\sqrt{\frac{p}{nq}}\Big) + p\exp\Big(it\sqrt{\frac{q}{np}}\Big)\Big)^n \\
&= \Big(1 - qit\sqrt{\frac{p}{nq}} - q\frac{t^2}{2}\frac{p}{nq} + pit\sqrt{\frac{q}{np}} - p\frac{t^2}{2}\frac{q}{np} + \frac{C_n}{n^{3/2}}\Big)^n \\
&= \Big(1 - \frac{t^2}{2}\frac{1}{n} + \frac{C_n}{n^{3/2}}\Big)^n \xrightarrow{n\to\infty} e^{-\frac{t^2}{2}} = \psi_{N(0,1)}(t).
\end{aligned}$$

Theorem 9.35

- Let $\mathbf{P}_1, \mathbf{P}_2, \dots \in \mathcal{P}(\mathbb{R}_+^d)$ and $\mathcal{L} : \mathbb{R}^d \to [0,1]$, so that $\mathcal{L}_{\mathbf{P}_n}(t) \xrightarrow{n\to\infty} \mathcal{L}(t)$ for all $t \in \mathbb{R}^d$.
  ($\mathcal{L}$ cont in 0) $\Rightarrow \mathbf{P}_n \xRightarrow{n\to\infty} \mathbf{P}$ for a $\mathbf{P} \in \mathcal{P}(\mathbb{R}^d)$ with $\mathcal{L}_{\mathbf{P}} = \mathcal{L}$.

Example: geometric $\Rightarrow$ exponential

- $X_n \sim \mu_{\text{geo}(p_n)}$ with $n \cdot p_n \xrightarrow{n\to\infty} \lambda$. Then

$$\begin{aligned}
\mathcal{L}_{X_n/n}(t) = \mathbf{P}[e^{-tX_n/n}] &= \sum_{k=1}^{\infty}(1-p_n)^{k-1}p_n e^{-tk/n} \\
&= p_n e^{-t/n}\frac{1}{1-(1-p_n)e^{-t/n}} \\
&= \frac{\lambda}{n(1-(1-\lambda/n)(1-t/n))} + o(1/n) \\
&\xrightarrow{n\to\infty} \frac{\lambda}{\lambda+t}.
\end{aligned}$$

This means $\frac{X_n}{n} \xRightarrow{n\to\infty} Y \sim \mu_{\exp(\lambda)}$, since

$$\mathcal{L}_{\exp(\lambda)}(t) = \int_0^\infty \lambda e^{-\lambda a}e^{-ta}da = \frac{\lambda}{\lambda+t}.$$