Probabily Theory
12. Separating classes of functions

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Separating points, separating

- Definition 9.20: $\mathcal{M} \subseteq \mathcal{C}(E)$

  separates points if

  $$\forall x \neq y \;\; \exists f \in \mathcal{M} : \quad f(x) \neq f(y).$$

  is separating in $\mathcal{P}(E)$ if for $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(E)$

  $$\forall x \in \mathcal{M} : \mathbf{P}[f] = \mathbf{Q}[f] \quad \Longrightarrow \quad \mathbf{P} = \mathbf{Q}.$$

- Example: $\mathcal{M} = \mathcal{C}_b(E)$ separates points and is separating.

  Indeed: $z \mapsto r(x,z) \wedge 1 \in \mathcal{C}_b(\mathbb{R})$ separates points. Let A be open and $f_n \uparrow 1_A$. If $\mathbf{P}[f_n] = \mathbf{Q}[f_n]$ then

  $$\mathbf{P}(A) = \lim_{n\to\infty} P[f_n] = \lim_{n\to\infty} Q[f_n] = \mathbf{Q}(A).$$

Algebra separating points → separating

- Theorem 9.24: (E, r) complete, separable. If $\mathcal{M} \subseteq \mathcal{C}_b(E)$ separates points and $f, g \in \mathcal{M} \Rightarrow$ also $fg \in \mathcal{M}$. Then $\mathcal{M}$ is separating.
- Theorem 9.23 (Stone-Weierstrass): Let (E, r) be compact and $\mathcal{M} \subseteq C_b(E)$ be an algebra which separates points, i.e. $1 \in \mathcal{M}$ and with $f, g \in \mathcal{M}$ and $\alpha, \beta \in \mathbb{R}$ is also $\alpha f + \beta g \in \mathcal{M}$. Then $\mathcal{M}$ is dense in $\mathcal{C}_b(E)$ with respect to the supremum norm.

Algebra separating points → separating

- Theorem 9.24: (E, r) complete, separable. If $\mathcal{M} \subseteq \mathcal{C}_b(E)$ separates points and $f, g \in \mathcal{M} \Rightarrow$ also $fg \in \mathcal{M}$. Then $\mathcal{M}$ is separating.

  Let $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(E)$; Let K compact with $\mathbf{P}(K^c) < \varepsilon$ and

  $C := \sup xe^{-x^2}$

  $$\left|\mathbf{P}[ge^{-\varepsilon g^2}] - \mathbf{P}[ge^{-\varepsilon g^2}; K]\right| \leq \frac{C}{\sqrt{\varepsilon}} \mathbf{P}(K^c) \leq C\sqrt{\varepsilon}$$

  Approximate $g_n \to g^{-\varepsilon g^2}$ on K with $g_n \in \mathcal{M}$.

$$\begin{aligned}\left|\mathbf{P}[g] - \mathbf{Q}[g]\right| &= \lim_{\varepsilon \to 0} \left|\mathbf{P}[ge^{-\varepsilon g^2}] - \mathbf{Q}[ge^{-\varepsilon g^2}]\right| \\ &\leq \left|\mathbf{P}[ge^{-\varepsilon g^2}] - \mathbf{P}[ge^{-\varepsilon g^2}; K]\right| + \cdots + \left|\mathbf{Q}[ge^{-\varepsilon g^2}; K] - \mathbf{Q}[ge^{-\varepsilon g^2}]\right| \\ &\leq 2C \lim_{\varepsilon \to 0} \sqrt{\varepsilon} = 0\end{aligned}$$

Characteristic function

- Proposition 9.25: $\mathbf{P} \in \mathcal{P}(\mathbb{R}^d)$ ($\mathbf{P} \in \mathcal{P}(\mathbb{R}^d_+)$) is uniquely given by $t \mapsto \psi_{\mathbf{P}}(t) := \mathbf{P}[e^{it\cdot}]$ ($\lambda \mapsto \mathcal{L}_{\mathbf{P}}(\lambda) := \mathbf{P}[e^{-\lambda\cdot}]$). $\mathcal{M} := \{x \mapsto e^{itx}; t \in \mathbb{R}^d\} \subseteq \mathcal{C}_b(\mathbb{R}^d)$. Furthermore, $\mathcal{M}$ is closed under multiplication and $1 \in \mathcal{M}$.

independence and characteristic function

- Corollary 9.26: $(X_j)_{j\in I}$ is independent if and only if if for all $J \subseteq_f I$

$$\mathbf{E}\Big[\prod_{j\in J} e^{it_j X_j}\Big] = \prod_{j\in J} \mathbf{E}[e^{it_j X_j}]$$

for all $(t_j)_{j\in J} \in \mathbb{R}^J$.