Probability Theory
11. Prohorov's Theorem

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Vague convergence

$$(\mathrm{P}_n \xrightarrow{n\to\infty}_v \mu) \qquad :\Longleftrightarrow \qquad \mathrm{P}_n[f] \xrightarrow{n\to\infty} \mu[f], \quad f \in \mathcal{C}_c(E)$$

- Lemma 9.12: Let $\mathrm{P}_1, \mathrm{P}_2, \dots \in \mathcal{P}(\mathbb{R})$ and μ measure on $\mathbb{R}$ with $\mathrm{P}_n[f] \xrightarrow{n\to\infty} \mu[f], f \in \mathcal{C}_c(\mathbb{R})$, then $\mu \in \mathcal{P}_{\leq 1}(\mathbb{R})$.

  Proof: Let $f_1, f_2, \dots \in \mathcal{C}_c(\mathbb{R})$ with $f_k \uparrow 1$. Then with monotone convergence

$$\mu(\mathbb{R}) = \sup_{k\in\mathbb{N}} \mu[f_k] = \sup_{k\in\mathbb{N}} \limsup_{n\to\infty} \mathrm{P}_n[f_k] \leq 1.$$

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Helly's Theorem

- Theorem 10.13: Let $\mathrm{P}_1, \mathrm{P}_2, \dots \in \mathcal{P}(\mathbb{R})$. Then there is a subsequence $(n_k)_{k=1,2,\dots}$ and a $\mu \in \mathcal{P}_{\leq 1}(\mathbb{R})$ with $\mathrm{P}_{n_k} \xrightarrow[]{k\to\infty}_v \mu$.
- In other words: The space $(\mathcal{P}_{\leq 1}, \to_v)$ is compact.

  Proof: F_n distribution function of P_n, and $(x_i)_{i=1,2,\dots}$ a denumeration of $\mathbb{Q}$. Choose a sequence $(n_k)_{k=1,2,\dots}$ and F right-continuous, non-increasing, so that $(F_{n_k}(x))_{k=1,2,\dots} \xrightarrow{k\to\infty} F(x)$ for all $x \in \mathbb{Q}$. Set $\mu((x,y]) := F(y) - F(x)$.

  To show: $\mathrm{P}_n[f] \xrightarrow{n\to\infty} \mu[f]$ for all $f \in \mathcal{C}_c(\mathbb{R})$. Wlog $f \geq 0$.

$$\mu[f] = \int_0^\infty \mu(f > t)dt \leq \liminf_{n\to\infty} \int_0^\infty \mathrm{P}_n(f > t)dt$$
$$= \liminf_{n\to\infty} \mathrm{P}_n[f] \leq \limsup_{n\to\infty} \mathrm{P}_n[f] = \dots = \mu[f].$$

Tightness

- Definition 9.14: Let $\mathcal{K} := \{A \subseteq E \text{ compact}\}$. A family $(\mathrm{P}_i)_{i\in I}$ in $\mathcal{P}(E)$ is *tight*, if

$$\sup_{K\in\mathcal{K}} \inf_{i\in I} \mathrm{P}_i(K) = 1.$$

$(X_i)_{i\in I}$ is called *tight* if $((X_i)_*\mathrm{P})_{i\in I}$ is tight, i.e.

$$\sup_{K\in\mathcal{K}} \inf_{i\in I} \mathrm{P}(X_i \in K) = 1.$$

- Equivalent formulation:
  $\forall\varepsilon > 0\ \exists K \text{ compact} : \inf_{i\in I} \mathrm{P}(X_i \in K) > 1-\varepsilon.$
- If E is Polish, then individual P are tight.
- If E is compact, then every family is tight.
- Let $(X_i)_{i\in I}$ be such that $\sup_{i\in I} \mathrm{P}[|X_i|] < \infty$. Then

$$\inf_{r>0} \sup_{i\in I} \mathrm{P}(|X_i| \geq r) \leq \inf_{r>0} \sup_{i\in I} \frac{\mathrm{P}[|X_i|]}{r} = 0.$$

Vague convergence and tightness

- Lemma 9.17: $\mu, \mathrm{P}_1, \mathrm{P}_2, \dots$ such that $\mathrm{P}_n \xrightarrow{n\to\infty}_v \mu$. Then

  $$(\mathrm{P}_n)_{n=1,2,\dots} \text{ tight} \qquad \iff \qquad \mu(\mathbb{R}) = 1.$$

  Then $\mathrm{P}_n \xRightarrow{n\to\infty} \mu$.

  For $r > 0$ let $g_r \in \mathcal{C}_c(\mathbb{R})$ with $1_{B_r(0)} \le g_r \le 1_{B_{r+1}(0)}$.

  $$(\mathrm{P}_n)_{n=1,2,\dots} \text{ tight} \qquad \iff \qquad \sup_{r>0} \liminf_{n\to\infty} \mathrm{P}_n[g_r] = 1.$$

  '$\Leftarrow$': $1 = \sup_{r>0} \mu[g_r] = \sup_{r>0} \liminf_{n\to\infty} \mathrm{P}_n[g_r] \le 1$.

  '$\Rightarrow$': $1 \ge \mu(\mathbb{R}) = \sup_{r>0} \mu[g_r] = \sup_{r>0} \liminf_{n\to\infty} \mathrm{P}_n[g_r] = 1$.

  to show: $\mathrm{P}_n \xRightarrow{n\to\infty} \mu$; let $f \in \mathcal{C}_b(\mathbb{R})$. Then

  $$\begin{aligned} |\mathrm{P}_n[f] - \mu[f]| &\le \big(|\mathrm{P}_n[f - fg_r]| + |\mathrm{P}_n[fg_r]| - \mu[fg_r]| + |\mu[f - fg_r]|\big) \\ &\le ||f||\mathrm{P}_n(B_r(0)^c) + |\mathrm{P}_n[fg_r]| - \mu[fg_r]| + \mu[B_r(0)^c], \end{aligned}$$

  from which $\mathrm{P}_n \xRightarrow{n\to\infty} \mu$ follows.

weak convergence and tightness

- Corollary 9.18: $\mathrm{P}, \mathrm{P}_1, \mathrm{P}_2, \dots \in \mathcal{P}(\mathbb{R})$.

  If $\mathrm{P}_n \xrightarrow[]{n\to\infty} \mathrm{P}$, then $(\mathrm{P}_n)_{n\in\mathbb{N}}$ is tight.

  Clear, since the vague convergence follows from weak convergence.

Prohorov's Theorem

- Theorem 9.19: (E, r) complete, separable and $(\mathrm{P}_i)_{i \in I}$ a family in $\mathcal{P}(E)$. Equivalent are:
  1. $(\mathrm{P}_i)_{i \in I}$ is relatively compact with respect to weak convergence.
  2. The family $(\mathrm{P}_i)_{i \in I}$ is tight.

  Proof for $I = \mathbb{N}$ and $E = \mathbb{R}$: $\Rightarrow$: Assume, $(\mathrm{P}_n)_{n=1,2,\dots}$ relatively compact but not tight. For $\varepsilon > 0$ choose TF so that $\mathrm{P}_{n_k}(B_k(0)^c) > \varepsilon$ and $\mathrm{P}_{n_k} \xrightarrow{k \to \infty} \mathrm{P}$. Since P tight, choose k such that

  $$\varepsilon > \mathrm{P}(B_k(0)^c) = \lim_m \mathrm{P}_{n_m}(B_k(0)^c) \geq \lim_m \mathrm{P}_{n_m}(B_m(0)^c) > \varepsilon$$

  $\Leftarrow$ Choose a vaguely convergent subsequence. Since this is tight, the statement follows with Lemma 9.17.