Probability Theory
10. Introduction to weak convergence

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Definition

- Preliminary remark: For the expected value with respect to P we now write

$$\mathrm{P}[X] := \int X d\mathrm{P}.$$

- $\mathcal{P}(E)$: probability measure on $\mathcal{B}(E)$;

  $\mathcal{P}_{\leq 1}(E)$: finite measure μ on $\mathcal{B}(E)$ with $\mu(E) \leq 1$.

  $\mathcal{C}_b(E)$: continuous, bounded functions $E \to \mathbb{R}$

  $\mathcal{C}_c(E)$: continuous functions $E \to \mathbb{R}$ with compact support

Definition 9.1

- Weak convergence of $\mathrm{P}_1, \mathrm{P}_2, \dots \in \mathcal{P}(E)$ to $\mathrm{P} \in \mathcal{P}(E)$:

$$\left(\mathrm{P}_n \xrightarrow{n\to\infty} \mathrm{P}\right) \quad :\iff \quad \mathrm{P}_n[f] \xrightarrow{n\to\infty} \mathrm{P}[f], \quad f \in \mathcal{C}_b(E)$$

- Vague convergence of $\mu_1, \mu_2, \dots \in \mathcal{P}_{\leq 1}$ to μ:

$$\left(\mu_n \xrightarrow{n\to\infty}_v \mu\right) \quad :\iff \quad \mu_n[f] \xrightarrow{n\to\infty} \mu[f], \quad f \in \mathcal{C}_c(E)$$

- Convergence in distribution of $X_1, X_2, \dots$ to X:

$$X_n \xRightarrow{n\to\infty} X \quad :\iff \quad (X_n)_*\mathrm{P}_n \xRightarrow{n\to\infty} X_*\mathrm{P}$$

$$\iff \quad \mathrm{P}[f(X_n)] \xrightarrow{n\to\infty} \mathrm{P}[f(X)], \quad f \in \mathcal{C}_b(E).$$

Examples

- $x, x_1, x_2, \dots \in \mathbb{R}$ with $x_n \xrightarrow{n\to\infty} x$ and

  $\mathrm{P} = \delta_x, \mathrm{P}_1 = \delta_{x_1}, \mathrm{P}_2 = \delta_{x_2}, \dots$ Then, $\mathrm{P}_n \xRightarrow{n\to\infty} \mathrm{P}$, since

  $$\mathrm{P}_n[f] = f(x_n) \xRightarrow{n\to\infty} f(x) = \mathrm{P}[f], \qquad f \in \mathcal{C}_b(\mathbb{R}).$$

  With $x_n = n$, we have $\mathrm{P}_n \xrightarrow{n\to\infty}_v 0$, since

  $$\mathrm{P}_n[f] = f(x_n) \xrightarrow{n\to\infty} 0 = 0[f], \qquad f \in \mathcal{C}_c(\mathbb{R}).$$

- Let $X, X_1, X_2, \dots$ identically distributed. Then $X_n \xRightarrow{n\to\infty} X$.
- Central Limit Theorem, for example as the theorem of deMoivre-Laplace: For $p \in (0,1)$ let

  $X_n \sim \mathrm{B}(n,p), n = 1, 2, \dots$ and $X \sim N(0,1)$. Then,

  $$\frac{X_n - np}{\sqrt{np(1-p)}} \xRightarrow{n\to\infty} X.$$

Uniqueness of the limit

- Lemma 9.4: Let $\mathrm{P}, \mathrm{Q}, \mathrm{P}_1, \mathrm{P}_2, \dots \in \mathcal{P}(E)$ with $\mathrm{P}_n \xrightarrow{n\to\infty} \mathrm{P}$ and $\mathrm{P}_n \xrightarrow{n\to\infty} \mathrm{Q}$. Then $\mathrm{P} = \mathrm{Q}$.

  Proof: to show: $\mathrm{P}(A) = \mathrm{Q}(A)$ for A closed. We set

  $$r(x, A) := \inf_{y \in A} r(x, y)$$

  and

  $$f_m(x) \mapsto (1 - m \cdot r(x, A))^+$$

  for $m = 1, 2, \dots$. Then $f_m \xrightarrow{m\to\infty} 1_A$. Further

  $$\mathrm{P}(A) = \lim_{m\to\infty} \mathrm{P}[f_m] = \lim_{m\to\infty} \lim_{n\to\infty} \mathrm{P}_n[f_m] = \lim_{m\to\infty} \mathrm{Q}[f_m] = \mathrm{Q}(A)$$

Convergence in probability and weak convergence

- Proposition 9.5: $X, X_1, X_2, \ldots$ rvs. If $X_n \xrightarrow{n\to\infty}_p X$, then $X_n \xRightarrow{n\to\infty} X$. If X is constant, then the inverse is also true.

  Proof: Suppose $X_n \xrightarrow{n\to\infty}_p X$ and $\lim_{n\to\infty} \mathrm{P}[f(X_n)] \neq \mathrm{P}[f(X)]$ for some $f \in \mathcal{C}_b(E)$. Choose a subsequence $(n_k)_{k=1,2,\ldots}$ and $\varepsilon > 0$ with $\lim_{k\to\infty} |\mathrm{P}[f(X_{n_k})] - \mathrm{P}[f(X)]| > \varepsilon$. Select a further subsequence $(n_{k_\ell})_{\ell=1,2,\ldots}$ with $X_{n_{k_\ell}} \xrightarrow{\ell\to\infty}_{as} X$. Then,

$$\lim_{\ell\to\infty} \mathrm{P}[f(X_{n_{k_\ell}})] = \mathrm{P}[f(X)]$$

Convergence in probability and weak convergence

- Proposition 9.5: $X, X_1, X_2, \ldots$ rvs. If $X_n \xrightarrow{n\to\infty}_p X$, then $X_n \xRightarrow{n\to\infty} X$. If X is constant, then the inverse is also true.

  Proof: weak convergence $\Rightarrow$ conv. in probability with $X = c$:

  Select $x \mapsto r(x, c) \wedge 1 \in \mathcal{C}_b(E)$, such that

  $$\mathrm{P}[r(X_n, c) \wedge 1] \xrightarrow{n\to\infty} \mathrm{P}[r(X, c) \wedge 1] = 0.$$

  From this, $X_n \xrightarrow{n\to\infty}_p X$ follows.

The Portmanteau Theorem

- Theorem 10.6: $X, X_1, X_2, \ldots$ rvs. Equivalent are:
  (i) $X_n \xRightarrow{n\to\infty} X$;
  (ii) $\mathrm{P}[f(X_n)] \xrightarrow{n\to\infty} \mathrm{P}[f(X)]$ for $f \in \mathcal{C}_b(E)$ Lipschitz;
  (iii) $\liminf_{n\to\infty} \mathrm{P}(X_n \in G) \geq \mathrm{P}(X \in G)$ for all open $G \subseteq E$.
  (iv) $\limsup_{n\to\infty} \mathrm{P}(X_n \in F) \leq \mathrm{P}(X \in F)$ for all closed $F \subseteq E$.
  (v) $\lim_{n\to\infty} \mathrm{P}(X_n \in B) = \mathrm{P}(X \in B)$ for all $B \in \mathcal{B}(E)$ with $\mathrm{P}(\partial B) = 0$.

  Proof: $(i) \Rightarrow (ii)$: clear; $(iii) \iff (iv)$ clear;

  $(ii) \Rightarrow (iv)$ $F \subseteq E$ closed, $\varepsilon_k \downarrow 0$ and

$$f_k(x) = \left(1 - \frac{1}{\varepsilon_k} r(x, F)\right)^+.$$

$$\limsup_{n\to\infty} \mathrm{P}(X_n \in F) \leq \limsup_{n\to\infty} \mathrm{P}[f_k(X_n)] = \mathrm{P}[f_k(X)] \downarrow \mathrm{P}(X \in F).$$

The Portmanteau Theorem

- Theorem 10.6: $X, X_1, X_2, \ldots$ rvs. Equivalent are:
  (i) $X_n \xRightarrow{n\to\infty} X$;
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  (iii) $\liminf_{n\to\infty} \mathrm{P}(X_n \in G) \geq \mathrm{P}(X \in G)$ for all open $G \subseteq E$.
  (iv) $\limsup_{n\to\infty} \mathrm{P}(X_n \in F) \leq \mathrm{P}(X \in F)$ for all closed $F \subseteq E$.
  (v) $\lim_{n\to\infty} \mathrm{P}(X_n \in B) = \mathrm{P}(X \in B)$ for all $B \in \mathcal{B}(E)$ with $\mathrm{P}(\partial B) = 0$.

Proof: $(iii) \Rightarrow (i)$ $f \in \mathcal{C}_b, 0 \leq f \leq c$:

$$\mathrm{P}[f(X)] = \int_0^\infty \mathrm{P}(f(X) > t)dt \leq \int_0^\infty \liminf_{n\to\infty} \mathrm{P}(f(X_n) > t)dt$$

$$\leq \liminf_{n\to\infty} \int_0^\infty \mathrm{P}(f(X_n) > t)dt = \liminf_{n\to\infty} \mathrm{P}[f(X_n)],$$

$$\limsup_{n\to\infty} \mathrm{P}[f(X_n)] = c - \liminf_{n\to\infty} \mathrm{P}[-f(X_n) + c] \leq \mathrm{P}[f(X)]$$

The Portmanteau Theorem

- 10.6: $X, X_1, X_2, \dots$ ZV. Equivalent are:
  - (i) $X_n \xRightarrow{n\to\infty} X$;
  - (ii) $\mathrm{P}[f(X_n)] \xrightarrow{n\to\infty} \mathrm{P}[f(X)]$ for $f \in \mathcal{C}_b(E)$ Lipschitz;
  - (iii) $\liminf_{n\to\infty} \mathrm{P}(X_n \in G) \geq \mathrm{P}(X \in G)$ for all open $G \subseteq E$.
  - (iv) $\limsup_{n\to\infty} \mathrm{P}(X_n \in F) \leq \mathrm{P}(X \in F)$ for all closed $F \subseteq E$.
  - (v) $\lim_{n\to\infty} \mathrm{P}(X_n \in B) = \mathrm{P}(X \in B)$ for all $B \in \mathcal{B}(E)$ with $\mathrm{P}(\partial B) = 0$.

$(iii), (iv) \Rightarrow (v)$ For $B \in \mathcal{B}(E)$ is

$$\mathrm{P}(X \in B^\circ) \leq \liminf_{n\to\infty} \mathrm{P}(X_n \in B^\circ) \leq \limsup_{n\to\infty} \mathrm{P}(X_n \in \overline{B}) \leq \mathrm{P}(X_n \in \overline{B}).$$

$(v) \Rightarrow (iv)$ For $F \subseteq E$ closed let $F^\varepsilon := \{x \in E : r(x, F) \leq \varepsilon\}$.
Then $\mathrm{P}(X \in \partial F^\varepsilon) = 0$ for Lebesgue-almost every ε. So,

$$\limsup_{n\to\infty} \mathrm{P}(X_n \in F) \leq \limsup_{n\to\infty} \mathrm{P}(X_n \in F^{\varepsilon_k}) = \mathrm{P}(X \in F^{\varepsilon_k}) \downarrow \mathrm{P}(X \in F).$$

Convergence of distribution functions

- Corollary 9.7: $\mathrm{P}, \mathrm{P}_1, \mathrm{P}_2, \dots \in \mathcal{P}(\mathbb{R})$ with distribution functions $F, F_1, F_2, \dots$ Then,

$$\mathrm{P}_n \xRightarrow{n\to\infty} \mathrm{P} \iff \Big(F_n(x) \xRightarrow{n\to\infty} F(x) \text{ for all continuity points } x \text{ of } F. \Big)$$

'$\Rightarrow$': If x is the continuity point of F, then $\mathrm{P}(\partial(-\infty; x]) = \mathrm{P}(\{x\}) = 0$. Also

$$F_n(x) = \mathrm{P}_n((-\infty; x]) \xrightarrow{n\to\infty} \mathrm{P}((-\infty; x]) = F(x).$$

'$\Leftarrow$': See manuscript;

The Theorem of deMoivre-Laplace

- For $X_n \sim B(n, p)$,

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq x\right) \xrightarrow{n\to\infty} \Phi(x),$$

where Φ is the distribution function of the standard normal distribution is.

This also means

$$\frac{X_n - np}{\sqrt{np(1-p)}} \xRightarrow{N\to\infty} Z \sim N(0, 1).$$

Slutzky's Theorem

- Corollary 9.9: $X, X_1, X_2, \dots, Y_1, Y_2, \dots$ rvs. If $X_n \xRightarrow{n\to\infty} X$ and $r(X_n, Y_n) \xRightarrow{n\to\infty}_p 0$, then $Y_n \xrightarrow{n\to\infty} X$.
- Proof: $f \in \mathcal{C}_b(E)$ Lipschitz. Then

$$|f(x) - f(y)| \leq L \cdot r(x,y) \wedge (2||f||_\infty), \qquad x, y \in E$$

$$\Rightarrow \limsup_{n\to\infty} |\mathsf{E}[f(X_n) - f(Y_n)]| \leq \limsup_{n\to\infty} \mathsf{E}[L \cdot r(X_n, Y_n) \wedge (2||f||_\infty)] = 0.$$

Also,

$$\limsup_{n\to\infty} \big|\mathsf{E}[f(Y_n)] - \mathsf{E}[f(X)]\big| \leq \limsup_{n\to\infty} \big|\mathsf{E}[f(Y_n)] - \mathsf{E}[f(X_n)]\big| + \big|\mathsf{E}[f(X_n)] - \mathsf{E}[f(X)]\big| = 0.$$

The Continuous Mapping Theorem

- Remark: $X, X_1, X_2, \dots$ ZV,

  $\varphi : E \to E'$ continuous. Then

  $$X_n \xrightarrow[]{n\to\infty} X \qquad \Rightarrow \qquad \varphi(X_n) \xrightarrow[]{n\to\infty} \varphi(X).$$

  Indeed: For $g \in \mathcal{C}_b(E')$ we have $g \circ \varphi \in \mathcal{C}_b(E)$, therefore

  $$\mathrm{P}[g(\varphi(X_n))] \xrightarrow{n\to\infty} \mathrm{P}[g(\varphi(X))].$$

- Theorem 9.10: $X, X_1, X_2, \dots$ ZV,

  $\varphi : E \to E'$ measurable,

  $U_\varphi := \{x : \varphi \text{ discontinuous in } x\} \subseteq E$.

  $$\mathrm{P}(X \in U_\varphi) = 0, \quad X_n \xrightarrow[]{n\to\infty} X \qquad \Rightarrow \qquad \varphi(X_n) \xrightarrow[]{n\to\infty} \varphi(X).$$

Weak and almost sure convergence, Skorohod

- Theorem 9.11: $X, X_1, X_2, \ldots$ ZV. Then, $X_n \overset{n\to\infty}{\Longrightarrow} X$ holds if and only if there is a probability space exists on which random variables $Y, Y_1, Y_2, \ldots$ are defined with $Y_n \xrightarrow{n\to\infty}_{as} Y$ and $Y \overset{d}{=} X, Y_1 \overset{d}{=} X_1, Y_2 \overset{d}{=} X_2, \ldots$
- Example: If $X, X_1, X_2, ...$ are iid, then $X_n \overset{n\to\infty}{\Longrightarrow} x$. With $X = Y_1, Y_2, ...$ we find $X_n \sim Y_n$ and $Y_n = X$, in particular $Y_n \xrightarrow{n\to\infty}_{as} X$.