Probability Theory
9. The Strong Law of Large Numbers

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The weak law of large numbers

- From basic probability:

  $X_1, X_2, \dots \in \mathcal{L}^2$ iid, $\varepsilon > 0$

$$\mathsf{P}\Big(\frac{1}{n}\Big|\sum_{k=1}^{n}(X_k - \mathsf{E}[X_k])\Big| > \varepsilon\Big) \leq \frac{1}{\varepsilon^2}\mathsf{V}\Big[\frac{1}{n}\sum_{k=1}^{n} X_k\Big] = \frac{1}{\varepsilon^2 n^2}\sum_{k=1}^{n}\mathsf{V}[X_k]$$
$$= \frac{\mathsf{V}[X_1]}{\varepsilon^2 n} \xrightarrow{n\to\infty} 0.$$

  In other words,

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - \mathsf{E}[X_k]) \xrightarrow{n\to\infty}_p 0.$$

Weak and strong law

- Definition 8.20: $X_1, X_2, \dots \in \mathcal{L}^1$ satisfies the weak law of large numbers if

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - \mathrm{E}[X_k]) \xrightarrow{n\to\infty}_p 0.$$

The sequence satisfies the strong law of large numbers if

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - \mathrm{E}[X_k]) \xrightarrow{n\to\infty}_{as} 0.$$

The strong law of large numbers

- Theorem 8.21: $X_1, X_2, \dots \in \mathcal{L}^1$ iid follows the strong law of large numbers, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{n\to\infty}_{as} \mathrm{E}[X_1].$$

Proof for $X_1 \in \mathcal{L}^4$: Wlog, let $\mathrm{E}[X_1] = 0$, $S_n = X_1 + \cdots + X_n$.

Since $\mathrm{E}[S_n/n] = \mathrm{E}[X_1]$ and by the Cauchy-Schwartz inequality,

$$\mathrm{E}[S_n^4] = \sum_{k=1}^{n} \mathrm{E}[X_k^4] + 6 \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathrm{E}[X_k^2 X_l^2] \leq (n + 6n^2)\mathrm{E}[X_1^4].$$

$$\text{So } \mathrm{E}\Big[\sum_{n=1}^{\infty} \Big(\frac{S_n}{n}\Big)^4\Big] \leq \sum_{n=1}^{\infty} \frac{n+6n^2}{n^4} \mathrm{E}[X_1^4] < \infty.$$

From $\sum_{n=1}^{\infty} \left(\frac{S_n}{n}\right)^4 < \infty$ we also find $\frac{S_n}{n} \xrightarrow{n\to\infty}_{as} 0$.

The Kronecker lemma

- Lemma 8.24: Let $x_1, x_2, \dots \in \mathbb{R}$, $y_1, y_2, \dots \in \mathbb{R}$ monotone with $y_n \uparrow \infty$ and $\sum_{n=1}^{\infty} x_n / y_n < \infty$. Then, $\sum_{k=1}^{n} x_k / y_n \xrightarrow{n\to\infty} 0$.

  Proof: Let $z_0 = 0, z_n := \sum_{k=1}^{n} x_k / y_k$. Then $z_n \xrightarrow{n\to\infty} z_\infty < \infty$ and $x_k = y_k(z_k - z_{k-1})$. With $y_0 = 0$,

$$\frac{\sum_{k=1}^{n} x_k}{y_n} = \frac{1}{y_n}\sum_{k=1}^{n} y_k(z_k - z_{k-1}) = z_n + \frac{1}{y_n}\Big(\sum_{k=0}^{n-1} y_k z_k - \sum_{k=1}^{n} y_k z_{k-1}\Big)$$

$$= z_n - \frac{1}{y_n}\Big(\sum_{k=1}^{n} y_k z_{k-1} - y_{k-1} z_{k-1}\Big)$$

$$\xrightarrow{n\to\infty} z_\infty - z_\infty \cdot \lim_{n\to\infty} \frac{1}{y_n}\sum_{k=1}^{n} y_k - y_{k-1} = 0.$$

The strong law of large numbers

- Theorem 9.21: $X_1, X_2, \dots \in \mathcal{L}^1$ iid follows the strong law of large numbers, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{n\to\infty}_{as} \mathrm{E}[X_1].$$

Proof for $X_1 \in \mathcal{L}^2$: Consider the sequence $X_1/1, X_2/2, \dots$

Since $\sum_{n=1}^{\infty} \mathrm{V}[X_n/n] = \mathrm{V}[X_1]\sum_{n=1}^{\infty} 1/n^2 < \infty$, we can use Theorem 8.19 that $\sum_{k=1}^{n} X_k/k$ converges almost surely. With Kronecker we then find $S_n/n \xrightarrow{n\to\infty}_{as} 0$.

The empirical distribution

- Definition 9.25: $X_1, X_2, \dots$ rvs. For $n = 1, 2, \dots,$ the (random) probability distribution

$$\widehat{\mu}_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{X_k}$$

is called the *empirical distribution* of $X_1, \dots, X_n$. If the random variables are real-valued, then

$$\widehat{F}_n(x) := \frac{1}{n} \sum_{k=1}^{n} 1_{X_k \leq x}$$

is the *empirical distribution function* of $X_1, \dots, X_n$.

The Glivenko-Cantelli theorem

- Theorem 8.26: $X_1, X_2, \ldots$ iid, real-valued, distribution function F. Then

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \widehat{F}_n(x) - F(x)| \xrightarrow{n\to\infty}_{as} 0.$$

Proof of pointwise convergence: For $x \in \mathbb{R}$,

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{k=1}^{n} 1_{X_k \le x}.$$

The statement follows because of

$\mathrm{E}[\widehat{F}_n(x)] = \mathrm{P}(X_1 \le x) = F(x)$ and the strong law.