

The background of the slide is a solid blue color with a large, faint watermark of the University of Vienna seal. The seal features a central figure, likely a scholar or saint, seated and holding a book. Above the figure are three smaller figures in niches. The entire scene is enclosed in a circular border with Latin text. The watermark is centered and serves as a background for the text.

Probability Theory

9. The Strong Law of Large Numbers

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The weak law of large numbers

- ▶ From basic probability:

$X_1, X_2, \dots \in \mathcal{L}^2$ iid, $\varepsilon > 0$

$$\begin{aligned} \mathbb{P}\left(\frac{1}{n} \left| \sum_{k=1}^n (X_k - \mathbb{E}[X_k]) \right| > \varepsilon\right) &\leq \frac{1}{\varepsilon^2} \mathbb{V}\left[\frac{1}{n} \sum_{k=1}^n X_k\right] = \frac{1}{\varepsilon^2 n^2} \sum_{k=1}^n \mathbb{V}[X_k] \\ &= \frac{\mathbb{V}[X_1]}{\varepsilon^2 n} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

In other words,

$$\frac{1}{n} \sum_{k=1}^n (X_k - \mathbb{E}[X_k]) \xrightarrow{n \rightarrow \infty} \mathbb{P} 0.$$

Weak and strong law

- ▶ Definition 8.20: $X_1, X_2, \dots \in \mathcal{L}^1$ satisfies the weak law of large numbers if

$$\frac{1}{n} \sum_{k=1}^n (X_k - \mathbb{E}[X_k]) \xrightarrow[n \rightarrow \infty]{p} 0.$$

The sequence satisfies the strong law of large numbers if

$$\frac{1}{n} \sum_{k=1}^n (X_k - \mathbb{E}[X_k]) \xrightarrow[n \rightarrow \infty]{as} 0.$$

The strong law of large numbers

- ▶ Theorem 8.21: $X_1, X_2, \dots \in \mathcal{L}^1$ iid follows the strong law of large numbers, i.e.

$$\frac{1}{n} \sum_{k=1}^n X_k \xrightarrow[n \rightarrow \infty]{as} E[X_1].$$

Proof for $X_1 \in \mathcal{L}^4$: Wlog, let $E[X_1] = 0$, $S_n = X_1 + \dots + X_n$.

Since $E[S_n/n] = E[X_1]$ and by the Cauchy-Schwartz inequality,

$$E[S_n^4] = \sum_{k=1}^n E[X_k^4] + 6 \sum_{\substack{k,l=1 \\ k \neq l}}^n E[X_k^2 X_l^2] \leq (n + 6n^2)E[X_1^4].$$

$$\text{So } E\left[\sum_{n=1}^{\infty} \left(\frac{S_n}{n}\right)^4\right] \leq \sum_{n=1}^{\infty} \frac{n + 6n^2}{n^4} E[X_1^4] < \infty.$$

From $\sum_{n=1}^{\infty} \left(\frac{S_n}{n}\right)^4 < \infty$ we also find $\frac{S_n}{n} \xrightarrow[n \rightarrow \infty]{as} 0$.

The Kronecker lemma

- Lemma 8.24: Let $x_1, x_2, \dots \in \mathbb{R}$, $y_1, y_2, \dots \in \mathbb{R}$ monotone with $y_n \uparrow \infty$ and $\sum_{n=1}^{\infty} x_n/y_n < \infty$. Then,

$$\sum_{k=1}^n x_k/y_n \xrightarrow{n \rightarrow \infty} 0.$$

Proof: Let $z_0 = 0$, $z_n := \sum_{k=1}^n x_k/y_k$. Then

$z_n \xrightarrow{n \rightarrow \infty} z_{\infty} < \infty$ and $x_k = y_k(z_k - z_{k-1})$. With $y_0 = 0$,

$$\begin{aligned} \frac{\sum_{k=1}^n x_k}{y_n} &= \frac{1}{y_n} \sum_{k=1}^n y_k(z_k - z_{k-1}) = z_n + \frac{1}{y_n} \left(\sum_{k=0}^{n-1} y_k z_k - \sum_{k=1}^n y_k z_{k-1} \right) \\ &= z_n - \frac{1}{y_n} \left(\sum_{k=1}^n y_k z_{k-1} - y_{k-1} z_{k-1} \right) \\ &\xrightarrow{n \rightarrow \infty} z_{\infty} - z_{\infty} \cdot \lim_{n \rightarrow \infty} \frac{1}{y_n} \sum_{k=1}^n y_k - y_{k-1} = 0. \end{aligned}$$

The strong law of large numbers

- ▶ Theorem 9.21: $X_1, X_2, \dots \in \mathcal{L}^1$ iid follows the strong law of large numbers, i.e.

$$\frac{1}{n} \sum_{k=1}^n X_k \xrightarrow[n \rightarrow \infty]{as} E[X_1].$$

Proof for $X_1 \in \mathcal{L}^2$: Consider the sequence $X_1/1, X_2/2, \dots$

Since $\sum_{n=1}^{\infty} V[X_n/n] = V[X_1] \sum_{n=1}^{\infty} 1/n^2 < \infty$, we can use

Theorem 8.19 that $\sum_{k=1}^n X_k/k$ converges almost surely. With

Kronecker we then find $S_n/n \xrightarrow[n \rightarrow \infty]{as} 0$.

The empirical distribution

- ▶ Definition 9.25: X_1, X_2, \dots rvs. For $n = 1, 2, \dots$, the (random) probability distribution

$$\hat{\mu}_n := \frac{1}{n} \sum_{k=1}^n \delta_{X_k}$$

is called the *empirical distribution* of X_1, \dots, X_n . If the random variables are real-valued, then

$$\hat{F}_n(x) := \frac{1}{n} \sum_{k=1}^n 1_{X_k \leq x}$$

is the *empirical distribution function* of X_1, \dots, X_n .

The Glivenko-Cantelli theorem

- ▶ Theorem 8.26: X_1, X_2, \dots iid, real-valued, distribution function F . Then

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| \xrightarrow[n \rightarrow \infty]{as} 0.$$

Proof of pointwise convergence: For $x \in \mathbb{R}$,

$$\hat{F}_n(x) = \frac{1}{n} \sum_{k=1}^n 1_{X_k \leq x}.$$

The statement follows because of

$E[\hat{F}_n(x)] = P(X_1 \leq x) = F(x)$ and the strong law.