Probability Theory
8. Sums of independent random variables

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Convolution of probability measures

- Definition 6.17: $\mathrm{P}_1, \ldots, \mathrm{P}_n$ probability measures on $\mathcal{B}(\mathbb{R})$ and $\mathrm{P}_1 \otimes \cdots \otimes \mathrm{P}_n$ their product measure. With $S(x_1, \ldots, x_n) := x_1 + \cdots + x_n$, the image measure

$$\mathrm{P}_1 * \cdots * \mathrm{P}_n := S_*(\mathrm{P}_1 \otimes \cdots \otimes \mathrm{P}_n)$$

is the *convolution* of $\mathrm{P}_1, \ldots, \mathrm{P}_n$

- Proposition 8.16: $X_1, \ldots, X_n$ independent. Then

$$(X_1 + \cdots + X_n)_* \mathrm{P} = (X_1)_* \mathrm{P} * \cdots * (X_n)_* \mathrm{P}$$

and $\psi_{X_1 + \cdots + X_n} = \psi_{X_1} \cdots \psi_{X_n}$ and $\mathcal{L}_{X_1 + \cdots + X_n} = \mathcal{L}_{X_1} \cdots \mathcal{L}_{X_n}$.

Proof: $(X_1, ..., X_n)_* \mathrm{P} = (X_1)_* \mathrm{P} \otimes \cdots \ldots (X_n)_* \mathrm{P}$

$$\mathrm{E}[e^{it(X_1 + \cdots + X_n)}] = \mathrm{E}\left[e^{itX_1} \cdots e^{itX_n}\right] = \mathrm{E}[e^{itX_1}] \cdots \mathrm{E}[e^{itX_n}]$$

Convergence of sums

- Prop. 8.17: $X_1, X_2, \dots$ independent, $S_n := X_1 + \cdots + X_n$.

$$P(\omega : S_n(\omega) \text{ converges as } n \to \infty) \in \{0, 1\},$$

$$P(\omega : S_n(\omega)/n \text{ converges as } n \to \infty) \in \{0, 1\}.$$

$(P(S_n/n \text{ converges}) = 1) \Rightarrow$ limit is constant, almost surely.

- Proof: $(\sigma(X_i))_{i=1,2,\dots}$ independent and

$$\{\omega : S_n(\omega) \text{ converges}\}, \{\omega : S_n(\omega)/n \text{ converges}\} \in \mathcal{T}.$$

$$S = \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \lim_{n\to\infty} \frac{X_m + \cdots + X_n}{n} \in \sigma\Big(\bigcup_{k\geq m} \mathcal{F}_k\Big)$$

Kolmogorov's maximal inequality of

- Prop. 8.18: $X_1, X_2, \dots \in \mathcal{L}^2$ independent, $K > 0$. Then .

$$P\Big(\sup_{n\in\mathbb{N}}\Big|\sum_{k=1}^{n} X_k - E[X_k]\Big| > K\Big) \le \frac{\sum_{n=1}^{\infty} V(X_n)}{K^2}.$$

Proof: Wlog $E[X_k] = 0$; $S_n = X_1 + \dots + X_n$,

$T := \inf\{n : |S_n| > K\} \Rightarrow P(\sup_n |S_n| > K) = P(T < \infty)$.

$$\begin{aligned}\sum_{k=1}^{n} E[X_k^2] = E[S_n^2] &\ge \sum_{k=1}^{n} E[S_n^2, T = k]\\ &= \sum_{k=1}^{n} E[S_k^2 + (S_n - S_k + 2S_k)(S_n - S_k), T = k]\\ &\ge \sum_{k=1}^{n} E[S_k^2, T = k] + 2E[S_k(S_n - S_k), T = k]\\ &= \sum_{k=1}^{n} E[S_k^2, T = k] \ge K^2 P(T \le n).\end{aligned}$$

Convergence criterion for series

- Thm 8.19: $X_1, X_2, \dots \in \mathcal{L}^2$ independent, $\sum_{n=1}^{\infty} \mathrm{V}[X_n] < \infty$.

  Then, $\sum_{k=1}^{n} X_k - \mathrm{E}[X_k]$ converges almost surely.

- Proof: Wlog, $\mathrm{E}[X_k] = 0$, $S_n = X_1 + \dots + X_n$. For $\varepsilon > 0$

  applies according to Proposition 8.18

$$\lim_{k\to\infty} \mathrm{P}(\sup_{n\geq k} |S_n - S_k| > \varepsilon) \leq \lim_{k\to\infty} \frac{\sum_{n=k+1}^{\infty} \mathrm{E}[X_n^2]}{\varepsilon^2} = 0.$$

  Therefore, $\sup_{n\geq k} |S_n - S_k| \xrightarrow{k\to\infty}_p 0$. According to

  Proposition 7.6, there is a subsequence $k_1, k_2, \dots$ with

  $\sup_{n\geq k_i} |S_n - S_{k_i}| \xrightarrow{i\to\infty}_{fs} 0$. However, since

  $(\sup_{n\geq k} |S_n - S_k|)_{k=1,2,\dots}$ is decreasing,

  $\sup_{n\geq k} |S_n - S_k| \xrightarrow{k\to\infty}_{fs} 0$.