Probability Theory
7. Kolmogorov's 0-1-Law

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Independence and generators

- Proposition 8.10: $(\mathcal{C}_i)_{i \in I}$ independent, $\cap$-stable set systems.

  Then, $(\sigma(\mathcal{C}_i))_{i \in I}$ are also independent.
- Recall: Let $\mathcal{C}$ be $\cap$-stable and $\mathcal{D} \supseteq \mathcal{C}$ Dynkin system

  $\Omega \in \mathcal{D}, \qquad A, B \in \mathcal{D}, A \subseteq B \Rightarrow B \setminus A \in \mathcal{D},$

  $A_1, A_2, \ldots \in \mathcal{D}, A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}.$

  Then $\sigma(\mathcal{C}) \subseteq \mathcal{D}$.
- Proof: Let $J = \{i_1, \ldots, i_n\} \subseteq_f I$ and o.E. $|J| > 1$. Then,

  $$\mathbf{P}(A_{i_1} \cap \cdots \cap A_{i_n}) = \prod_{k=1}^{n} \mathbf{P}(A_{i_k}) \text{ for } A_{i_k} \in \mathcal{C}_{i_k}, k = 1, \ldots, n. \qquad (*)$$

  Fix $A_{i_2}, \ldots, A_{i_n}$ and show that $\mathcal{D} := \{A_{i_1} \in \mathcal{F} : (*) \text{ holds}\}$. is a $\cap$-stable Dynkin system.

Indicator functions

- Corollary 8.11: A family of sets $(A_i)_{i\in I}$ is independent if and only if the family of random variables $(1_{A_i})_{i\in I}$ is independent. In particular,

$$\mathbf{P}\Big(\bigcap_{j\in J} B_j\Big) = \prod_{j\in J} \mathbf{P}(B_j)$$

for $J \subseteq_f I$, $B_j \in \{A_j, A_j^c\}, j \in J$.

Proof: $\mathcal{C}_i = \{A_i\}$, $\sigma(\mathcal{C}_i) = \{\emptyset, C_i, C_i^c, \Omega\}$

Grouping

- Corollary 8.12: $(\mathcal{F}_i)_{i\in I}$ Family of independent σ-algebras, $\mathcal{I}$ a partition of I, i.e. $\mathcal{I} = \{I_k, k \in K\}$ with $\biguplus_{k\in K} I_k = I$. Then $(\sigma(\mathcal{F}_i : i \in I_k))_{k\in K}$ is also an independent system.
  Proof: $\mathcal{C}_k := \left\{ \bigcap_{i\in J_k} A_i : J_k \subseteq_f I_k, A_i \in \mathcal{F}_i \right\}$ is $\cap$-stable and $\sigma(\mathcal{C}_k) = \sigma(\mathcal{F}_i : i \in I_k)$, $k \in K$.

Terminal σ-algebra

- Definition 8.13: Let $\mathcal{F}_1, \mathcal{F}_2, ... \subseteq \mathcal{F}$ all σ-algebras. Then

$$\mathcal{T}(\mathcal{F}_1, \mathcal{F}_2, \dots) = \bigcap_{n \geq 1} \sigma\Big(\bigcup_{m > n} \mathcal{F}_m \Big)$$

is the σ*-algebra of the terminal events* of $\mathcal{F}_1, \mathcal{F}_2, \dots$ $\widetilde{\mathcal{F}} \subseteq \mathcal{F}$

A σ-algebra $\widetilde{\mathcal{F}}$ is called **P**-trivial if $\mathbf{P}(A) \in \{0, 1\}$, $A \in \widetilde{\mathcal{F}}$.

Trivial σ-algebras

- Lemma 8.14:

  $\widetilde{\mathcal{F}}$ σ-algebra is $\mathbf{P}$-trivial $\iff$ $\widetilde{\mathcal{F}}$ is independent of itself.

  $\widetilde{\mathcal{F}}$ is a $\mathbf{P}$-trivial σ-algebra, X is $\widetilde{\mathcal{F}}$-measurable. Then X is constant, almost surely.

  Proof: '$\Rightarrow$':

  $A, B \in \widetilde{\mathcal{F}} \Rightarrow \mathbf{P}(A \cap B) = \mathbf{P}(A) \wedge \mathbf{P}(B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$, so $\widetilde{\mathcal{F}}$ is independent of itself.

  '$\Leftarrow$': For $A \in \widetilde{\mathcal{F}}$,

  $\mathbf{P}(A) = \mathbf{P}(A \cap A) = \mathbf{P}(A)^2 \Rightarrow \mathbf{P}(A) \in \{0, 1\}$.

  Let $c := \sup\{x : \mathbf{P}(X < x) = 0\}$, thus

  $1 = \lim_{\varepsilon \downarrow 0} \mathbf{P}(X < c + \varepsilon) - \mathbf{P}(X < c - \varepsilon) = P(X = c)$

Kolmogorov's 0-1 law

- Theorem 8.15: Let $(\mathcal{F}_n)_{n=1,2,\dots}$ be independent σ-algebras.

  Then, $\mathcal{T} := \mathcal{T}(\mathcal{F}_1, \mathcal{F}_2, \dots)$ is $\mathbf{P}$-trivial.

  Proof:

$$\mathcal{T}_n := \sigma\Big(\bigcup_{m>n} \mathcal{F}_m\Big) \qquad n = 1, 2, \dots .$$

  According to Corollary 8.12: $(\mathcal{F}_1, \dots, \mathcal{F}_n, \mathcal{T}_n)$ independent, $n = 1, 2, \dots$ This also means that $(\mathcal{F}_1, \dots, \mathcal{F}_n, \mathcal{T})$ are independent, $n = 1, 2, \dots$ and therefore also $(\mathcal{T}, \mathcal{F}_1, \mathcal{F}_2, \dots)$. Again with Corollary 8.12 it follows that $(\mathcal{T}_0, \mathcal{T})$ are independent and, since $\mathcal{T} \subseteq \mathcal{T}_0$, it also follows that $\mathcal{T}$ is independent of itself. Therefore, the assertion follows from Lemma 8.14.