Probability Theory
6. The Lemma of Borel-Cantelli

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May 5, 2024

Independence

- Definition 8.1: $(A_i)_{i\in I}$ with $A_i \in \mathcal{F}$ is called *independent*, if

$$\mathrm{P}\Big(\bigcap_{j\in J} A_j\Big) = \prod_{j\in J} \mathrm{P}(A_j) \qquad (*)$$

for all $J \subseteq_f I$.

$(\mathcal{C}_i)_{i\in I}$ with $\mathcal{C}_i \subseteq \mathcal{F}$ is called *independent* if $(*)$ holds for all $J \subseteq_f I$ and $A_j \in \mathcal{C}_j, j \in J$.

$(X_i)_{i\in I}$ is called independent if $(\sigma(X_i))_{i\in I}$ is independent.

Existence of product measures

- Proposition 8.2: $(X_i)_{i\in I}$ is independent if and only if for each $J \subseteq_f I$
$$((X_i)_{i\in J})_* \mathrm{P} = \bigotimes_{i\in J} (X_i)_* \mathrm{P},$$
- Corollary 8.3: Let E be Polish, I arbitrary. Let $X_i : \Omega : i \to E$ ZV for probability spaces $(\Omega_i, \mathcal{F}_i, \mathrm{P}_i)$, $i \in I$. Then there are $(\Omega, \mathcal{F}, \mathrm{P})$ and $Y_i : \Omega \to E$ with $(Y_i)_{i\in I}$ independent and $Y_i \stackrel{d}{=} X_i$.
- Lemma 8.4: $(\Omega_i', \mathcal{F}_i'), (\Omega_i'', \mathcal{F}_i'')$, $i \in I$, measureable spaces. $(X_i)_{i\in I}$ independent rvs, $X_i : \Omega \to \Omega_i'$, and $\varphi_i : \Omega_i' \to \Omega_i''$ measurable, $i \in I$. Then the family $(\varphi_i(X_i))_{i\in I}$ is also independent. Proof: Clear because of $\sigma(\varphi_i(X_i)) \subseteq \sigma(X_i)$.

Independent and uncorrelated

- Proposition 8.5: $X, Y \in \mathcal{L}^1$ independent, $\mathbb{R}$-valued. Then $XY \in \mathcal{L}^1$ and

$$\mathrm{E}[XY] = \mathrm{E}[X] \cdot \mathrm{E}[Y].$$

Proof: If the statement is true, it is also true for sums:

$$\mathrm{E}\Big[\sum_{i=1}^n X_i \cdot \sum_{j=1}^n Y_j\Big] = \sum_{i=1}^n \sum_{j=1}^n \mathrm{E}[X_i Y_j] = \sum_{i=1}^n \sum_{j=1}^n \mathrm{E}[X_i]\mathrm{E}[Y_j]$$
$$= \mathrm{E}\Big[\sum_{i=1}^n X_i\Big] \cdot \mathrm{E}\Big[\sum_{j=1}^n Y_j\Big].$$

Clear if $X = 1_A, Y = 1_B$;

Clear for simple functions

Clear for non-negative, measurable functions

Example:

- Let $X, Y \sim B(1, .5)$. Then $X+Y, X-Y$ are uncorrelated but not independent.

$$E[(X+Y)(X-Y)] = E[X^2 - Y^2] = E[X^2] - E[Y^2] = 0,$$

but

$$\begin{aligned} P(X+Y=2, X-Y=1) &= 0 \\ &\neq \tfrac{1}{16} = P(X=Y=1)\cdot P(X=1, Y=0) \\ &= P(X+Y=2)\cdot P(X-Y=1). \end{aligned}$$

The Borel-Cantelli lemma

- Definition 8.7: For $A_1, A_2, \dots \in \mathcal{F}$,

$$\limsup_{n\to\infty} A_n := \bigcap_{n\geq 1} \bigcup_{m\geq n} A_m$$

.

- Theorem 8.8: Let $A_1, A_2, ... \in \mathcal{F}$. Then

$$\sum_{n=1}^{\infty} \mathrm{P}(A_n) < \infty \Longrightarrow \mathrm{P}(\limsup_{n\to\infty} A_n) = 0.$$

If $A_1, A_2, \dots$ are independent,

$$\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \infty \Longrightarrow \mathrm{P}(\limsup_{n\to\infty} A_n) = 1.$$

Proof:

$$\mathrm{P}(\limsup_{n\to\infty} A_n) = \lim_{n\to\infty} \mathrm{P}\Big(\bigcup_{m\geq n} A_m\Big) \leq \lim_{n\to\infty} \sum_{m=n}^{\infty} \mathrm{P}(A_m) = 0$$

The Borel-Cantelli lemma

- Theorem 8.8: Let $A_1, A_2, ... \in \mathcal{F}$. If $A_1, A_2, \ldots$ are independent,

$$\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \infty \Longrightarrow \mathrm{P}(\limsup_{n\to\infty} A_n) = 1.$$

Proof: We recall $\log(1-x) \leq -x$ for $x \in [0,1]$.

$$\mathrm{P}((\limsup_{n\to\infty} A_n)^c) = \mathrm{P}\Big(\bigcup_{n=1}^{\infty} \bigcap_{m\geq n} A_m^c\Big) = \lim_{n\to\infty} \mathrm{P}\Big(\bigcap_{m=n}^{\infty} A_m^c\Big)$$

$$= \lim_{n\to\infty} \prod_{m=n}^{\infty} (1 - \mathrm{P}(A_m)) = \lim_{n\to\infty} \exp\Big(\sum_{m=n}^{\infty} \log(1 - \mathrm{P}(A_m))\Big)$$

$$\leq \lim_{n\to\infty} \exp\Big(-\sum_{m=n}^{\infty} \mathrm{P}(A_m)\Big)$$

$$= 0$$

Examples

- Let $X = (X_1, X_2, \dots)$ be an infinite p coin toss with $p > 0$.

  Then $|\{n : X_n \text{ head}\}| = \infty.$ head.

  Indeed: Let $A_n := \{X_n \text{ head}\}$. Then, $A_1, A_2, \dots$ is independent

  and $\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \sum_{n=1}^{\infty} \frac{1}{2} = \infty$.
- The probability that only finitely many of the events

  $B_n := \{X_1 \text{ head}\}$ occur is p.
- Let $X_n \sim \mathrm{geo}(p)$ be independent. Then, $|\{n : X_n > n\}| < \infty$

  almost surely.

  Indeed, let $A_n := \{X_n > n\}$. Then,

  $$\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \sum_{n=1}^{\infty} \mathrm{P}(X_n \geq n) = \sum_{n=1}^{\infty} (1-p)^{n-1} = \frac{1}{p} < \infty.$$