Probability Theory
5. Convergence in probability and $\mathcal{L}^p$-convergence

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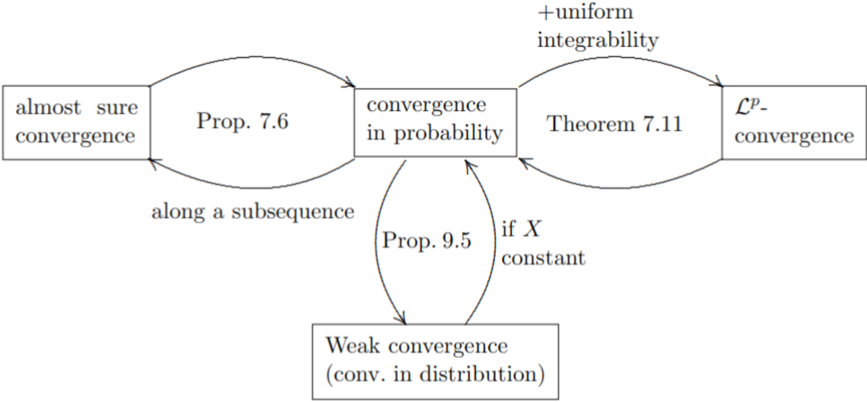
Kinds of convergence



Uniform integrability

Let $U \sim U([0,1])$.

- $(Y_n \xrightarrow{n\to\infty}_{as} Y) \not\to (Y_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} Y)$

  with $Y = 0$ and $Y_n := n \cdot 1_{U \in B_n}$ for $B_n = [0, \frac{1}{n}]$. Here,

$$P(\lim_{n\to\infty} Y_n = 0) = P(U > 0) = 1,$$

  i.e. $Y_n \xrightarrow{n\to\infty}_{as} 0$, but is $E[Y_n] = E[Y_n - 0] = 1 \neq 0$.

- Definition 7.7: $(X_i)_{i\in I}$ is *uniformly integrable*, if

$$\inf_K \sup_{i\in I} E[|X_i|; |X_i| > K] = 0$$

- For $(Y_n)_{n=1,2,\dots}$ as above is

$$\inf_K \sup_{n=1,2,\dots} E[|Y_n|; |X_n| > K] = \inf_K \sup_{n>K} E[|Y_n|] = 1.$$

Examples

Let $(X_i)_{i\in I}$ be a family of rvs.

- Let $Y \in \mathcal{L}^1$ and $|X_i| \le Y, i \in I$. Then, $(X_i)_{i\in I}$ is uniformly integrable:

$$\sup_{i\in I} \mathsf{E}[|X_i|; |X_i| > K] \le \mathsf{E}[|Y|; |Y| > K] \xrightarrow{K\to\infty} 0$$

- If I is finite and $X_i \in \mathcal{L}^1$, then $(X_i)_{i\in I}$ is uniformly integrable:

$$S := \sum_i |X_i| \in \mathcal{L}^1 \Rightarrow \sup_{1\le i\le n} \mathsf{E}[|X_i|; |X_i| > K] \le \mathsf{E}[S; S > K] \to 0$$

- $X_i \in \mathcal{L}^p$ for $p > 1$ and $\sup_{i\in I} \mathsf{E}[|X_i|^p] < \infty$. Then $(X_i)_{i\in I}$ is uniformly integrable:

$$\sup_{i\in I} \mathsf{E}[|X_i|; |X_i| > K] \le \sup_{i\in I} \frac{\mathsf{E}[|X_i|^p]}{K^{p-1}} \xrightarrow{K\to\infty} 0.$$

Characterization of uniform integrability

- Lemma 7.9: For $(X_i)_{i\in I}$, the following are equivalent:
  1. $(X_i)_{i\in I}$ uniformly integrable.
  2. $\sup_{i\in I} \mathrm{E}[|X_i|] < \infty$ and $\lim_{\varepsilon\to 0} \sup_{A:\mathrm{P}(A)<\varepsilon} \sup_{i\in I} \mathrm{E}[|X_i|; A] = 0$,
  3. $\lim_{K\to\infty} \sup_{i\in I} \mathrm{E}[(|X_i| - K)^+] = 0$.
  4. There is $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\frac{f(x)}{x} \xrightarrow{x\to\infty} \infty$ and $\sup_{i\in I} \mathrm{E}[f(|X_i|)] < \infty$.

  In any of these cases, f in 4. can be chosen to be monotonically increasing and convex.

  1.⇒2.: $\delta, K > 0$ such that $\sup_{i\in I} \mathrm{E}[|X_i|; |X_i| > K] \le \delta$. Then,

$$\mathrm{E}[|X_i|; A] = \mathrm{E}[|X_i|; A\cap\{|X_i| > K\}] + \mathrm{E}[|X_i|; A\cap\{|X_i| \le K\}] \le \delta + K\cdot\mathrm{P}(A).$$

$$\sup_{i\in I} \mathrm{E}[|X_i|] = \sup_{i\in I} \mathrm{E}[|X_i|; \Omega] \le \delta + K < \infty$$

$$\sup_{A:\mathrm{P}(A)<\varepsilon} \sup_{i\in I} \mathrm{E}[|X_i|; A] \le \delta + K\varepsilon \xrightarrow{\varepsilon\to 0} \delta.$$

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  3. $\lim_{K\to\infty}\sup_{i\in I}\mathrm{E}[(|X_i|-K)^+] = 0$.
  4. There is $f:\mathbb{R}_+\to\mathbb{R}_+$ such that $\frac{f(x)}{x}\xrightarrow{x\to\infty}\infty$ and $\sup_{i\in I}\mathrm{E}[f(|X_i|)]<\infty$.

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  2.⇒3.: $(|X_i|-K)^+ \le |X_i|1_{|X_i|\ge K}$. Let $\mathrm{P}(|X_i|>K_\varepsilon)<\varepsilon$

$$\lim_{K\to\infty}\sup_{i\in I}\mathrm{E}[(|X_i|-K)^+] = \lim_{\varepsilon\to 0}\sup_{i\in I}\mathrm{E}[(|X_i|-K_\varepsilon)^+]$$
$$\le \lim_{\varepsilon\to 0}\sup_{i\in I}\mathrm{E}[|X_i|;|X_i|>K_\varepsilon] \le \lim_{\varepsilon\to 0}\sup_{A:\mathrm{P}(A)<\varepsilon}\sup_{i\in I}\mathrm{E}[|X_i|;A] = 0.$$

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  3.→4.: Let $K_n\uparrow\infty$ with $\sup_{i\in I}\mathrm{E}[(|X_i|-K_n)^+]\le 2^{-n}$ and

$$f(x) := \sum_{n=1}^\infty (x-K_n)^+ \text{ monotonically increasing, convex}$$

$$x\ge 2K_n: \frac{f(x)}{x} \ge \sum_{k=1}^n \Big(1-\frac{K_k}{x}\Big) \ge \frac{n}{2},$$

$$\mathrm{E}[f(|X_i|)] = \sum_{n=1}^\infty \mathrm{E}[(|X_i|-K_n)^+] \le \sum_{n=1}^\infty 2^{-n} = 1.$$

Characterization of uniform integrability

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  2. $\sup_{i\in I} \mathrm{E}[|X_i|] < \infty$ and
     $\lim_{\varepsilon\to 0} \sup_{A:\mathrm{P}(A)<\varepsilon} \sup_{i\in I} \mathrm{E}[|X_i|; A] = 0$,
  3. $\lim_{K\to\infty} \sup_{i\in I} \mathrm{E}[(|X_i| - K)^+] = 0$.
  4. There is $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\frac{f(x)}{x} \xrightarrow{x\to\infty} \infty$ and
     $\sup_{i\in I} \mathrm{E}[f(|X_i|)] < \infty$.

  In any of these cases, f in 4. can be chosen to be monotonically increasing and convex.

  4.→1.: For $a_K := \inf_{x\geq K} \frac{f(x)}{x}$, so that also $a_K \xrightarrow{K\to\infty} \infty$,

$$\begin{aligned}\sup_{i\in I} \mathrm{E}[|X_i|; |X_i| \geq K] &\leq \frac{1}{a_K} \sup_{i\in I} \mathrm{E}[f(|X_i|); |X_i| \geq K]\\ &\leq \frac{1}{a_K} \sup_{i\in I} \mathrm{E}[f(|X_i|)] \xrightarrow{K\to\infty} 0.\end{aligned}$$

Sum and uniform integrability

- Let $X \in \mathcal{L}^p$ with $p \geq 1$. Then

  $(|X_i|^p)_{i \in I}$ is unif. integrable $\iff$ $|X_i + X|^p_{i \in I}$ unif. integrable.

  Indeed:

  $$\sup_{i \in I} \mathrm{E}[|X_i + X|^p]^{1/p} \leq \mathrm{E}[|X|^p]^{1/p} + \sup_{i \in I} \mathrm{E}[|X_i|^p]^{1/p} < \infty$$

  and

  $$\sup_{A:\mathrm{P}(A)<\varepsilon} \sup_{i \in I} \mathrm{E}[|X_i + X|^p; A]^{1/p}$$
  $$\leq \sup_{A:\mathrm{P}(A)<\varepsilon} \sup_{i \in I} \mathrm{E}[|X_i|^p; A]^{1/p} + \sup_{A:\mathrm{P}(A)<\varepsilon} \mathrm{E}[|X|^p; A]^{1/p} \xrightarrow{\varepsilon \to 0} 0,$$

Convergence in probability and $\mathcal{L}^p$-convergence

- Theorem 7.11: $X_1, X_2, \dots \in \mathcal{L}^p$. Then, (There is $X \in \mathcal{L}^p$ with $X_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} X$) $\iff$ $((|X_n|^p)_{n=1,2,\dots}$ is uniformly integrable and there is X with $X_n \xrightarrow{n\to\infty}_p X$.) In any case, the limits match.

  $\Rightarrow$:

  $$P(|X_n - X| > \varepsilon) \leq \frac{E[|X_n - X|^p]}{\varepsilon^p} \xrightarrow{n\to\infty} 0,$$

  to show: $(|X_i - X|^p)_{i\in I}$ uniformly interable;

  $\sup_{n=1,2,\dots} E[|X_n - X|^p] < \infty$

  $$\sup_{A:P(A)<\varepsilon} \sup_{n\in\mathbb{N}} (E[|X_n - X|^p; A])$$

  $$\leq \sup_{A:P(A)<\varepsilon} \sup_{n=1,\dots,N} (E[|X_n - X|^p; A]) + \sup_{n>N} (E[|X_n - X|^p])$$

Convergence in probability and $\mathcal{L}^p$-convergence

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  $\Leftarrow$: $\mathrm{E}[|X|^p] = \mathrm{E}[\liminf_{k\to\infty} |X_{n_k}|^p] \le \sup_{n\in\mathbb{N}} \mathrm{E}[|X_n|^p] < \infty$

$$\begin{aligned} &\mathrm{E}[|X_n - X|^p] \\ &\quad \le \mathrm{E}[|X_n - X|^p; |X_n - X| > \delta] + \mathrm{E}[|X_n - X|^p \wedge 1] \\ &\quad \xrightarrow{\delta\to 0} \mathrm{E}[|X_n - X|^p \wedge 1] \xrightarrow{n\to\infty} 0 \end{aligned}$$

Expectation and uniform integrability

- Corollary 7.12: Let $X_n \xrightarrow{n\to\infty}_p X$. Equivalent are:
  1. $X_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} X$,
  2. $||X_n||_p \xrightarrow{n\to\infty} ||X||_p$,
  3. The family $(|X_n|^p)_{n=1,2,\dots}$ is uniformly integrable.

1. $\Leftrightarrow$ 3. clear from Theorem 7.11

1. $\Rightarrow$ 2.:

$$\big| ||X_n|_p - ||X||_p \big| \le ||X_n - X|_p \xrightarrow{n\to\infty} 0.$$

2. $\Rightarrow$ 3.:

$$\begin{aligned} E[|X_n|^p; |X_n| > K] &\le E[|X_n|^p - (|X_n| \wedge (K - |X_n|)^+)^p] \\ &\xrightarrow{n\to\infty} E[|X|^p - (|X| \wedge (K - |X|)^+)^p] \\ &\xrightarrow{K\to\infty} 0 \end{aligned}$$