Probability Theory
2. Moments, characteristic functions and Laplace transforms

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April 23, 2024

Moments

- Definition 6.8: X, Y real-valued RVs. If it exists, $\mathbf{E}[X]$ is called *expected value* of X and

$$\mathbf{V}[X] := \mathbf{E}[(X - \mathbf{E}[X])^2]$$

variance of X and

$$\mathbf{COV}[X, Y] := \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$$

covariance of X and Y.

If $\mathbf{COV}[X, Y] = 0$, then X and Y are called uncorrelated. For $p > 0$, $\mathbf{E}[X^p]$ is the p-th moment of X and $\mathbf{E}[(X - \mathbf{E}[X])^p]$ is the centered p-th moment of X.

- $\mathcal{L}^p := \mathcal{L}^p(\mathbf{P}) := \{X : \mathbf{E}[X^p] \text{ exists.}\}$

Properties of the second moments

- Proposition 6.9: $X, Y \in \mathcal{L}^2$. Then,

  $\mathbf{V}[X], \mathbf{V}[Y], \mathbf{COV}[X, Y] < \infty$ and

  $$\mathbf{V}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2,$$

  $$\mathbf{COV}[X, Y] = \mathbf{E}[XY] - \mathbf{E}[X] \cdot \mathbf{E}[Y].$$

  The Cauchy-Schwarz inequality is

  $$\mathbf{COV}[X, Y]^2 \leq \mathbf{V}[X] \cdot \mathbf{V}[Y].$$

  If $X_1, ; X_n \in \mathcal{L}^2$, then the equation of Bienamyé is

  $$\mathbf{V}\Big[\sum_{k=1}^{n} X_k\Big] = \sum_{k=1}^{n} \mathbf{V}[X_k] + 2 \sum_{1 \leq k < l \leq n} \mathbf{COV}[X_k, X_l].$$

Alternative calculation of $\mathbb{E}[X^p]$

- Proposition 6.10: $X \geq 0$ ZV. Then applies

$$\mathbf{E}[X^p] = p\int_0^\infty \mathbf{P}(X > t)t^{p-1}dt = p\int_0^\infty \mathbf{P}(X \geq t)t^{p-1}dt.$$

Proof: Fubini:

$$\mathbf{E}[X^p] = p\mathbf{E}\Big[\int_0^X t^{p-1}dt\Big] = p\int_0^\infty \mathbf{E}\Big[1_{X>t}t^{p-1}\Big]dt$$

$$= p\int_0^\infty \mathbf{P}(X > t)t^{p-1}dt.$$

Second equation analogous

Characteristic functions, Laplace transform

- Definition 6.11: Let X be $\mathbb{R}^d$-valued RV. The *characteristic function of X* is

$$\psi_X(t) := \psi_{X_*\mathbf{P}}(t) := \mathbf{E}[e^{itX}] := \mathbf{E}[\cos(tX)] + i\mathbf{E}[\sin(tX)],$$

where $tx := \langle t, x \rangle$ is the scalar product.

The *Laplace transform* of X is

$$\mathcal{L}_X(t) := \mathcal{L}_{X_*\mathbf{P}}(t) := \mathbf{E}[e^{-tX}],$$

if the right-hand side exists.

Properties of the characteristic functions

- Proposition 6.12: X, Y ZV with values in $\mathbb{R}^d$. Then,

$$|\psi_X(t)| \le 1, \qquad \psi_X(0) = 1,$$

$$\psi_X \text{ is uniformly continuous}, \quad \psi_{aX+b}(t) = \psi_X(at)e^{ibt}.$$

Proof of uniform continuity. First of all

$$\begin{aligned} |e^{ihx} - 1| &= \sqrt{|\cos(hx) + i\sin(hx) - 1|^2} \\ &= \sqrt{(\cos(hx) - 1)^2 + \sin(hx)^2} \\ &= \sqrt{2(1 - \cos(hx))} = 2|\sin(hx/2)| \le |hx| \wedge 2, \end{aligned}$$

$$\begin{aligned} sup_{t\in\mathbb{R}^d}|\psi_X(t+h) - \psi_X(t)| &= \sup_{t\in\mathbb{R}^d} |\mathbf{E}[e^{i(t+h)X} - e^{itX}]| \\ &= \sup_{t\in\mathbb{R}^d} |\mathbf{E}[e^{itX}(e^{ihX} - 1)]| \le \mathbf{E}[|e^{ihX} - 1|] \le \mathbf{E}[|hX| \wedge 2] \\ &\xrightarrow{h\to 0} 0. \end{aligned}$$

Example: binomial distribution, Poisson distribution

- Let $X \sim B(n,p)$ be. Then

$$\psi_{B(n,p)}(t) = \mathbf{E}[e^{itX}] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{itk} = (1-p+pe^{it})^n.$$

- Let $X \sim \mathrm{Poi}(\gamma)$. Then is

$$\psi_{\mathrm{Poi}(\gamma)}(t) = \mathbf{E}[e^{itX}] = e^{-\gamma} \sum_{n=0}^{\infty} \frac{\gamma^n e^{itn}}{n!} = e^{\gamma(e^{it}-1)}.$$

Example: Normal distribution, exponential distribution

- Let $X \sim N(\mu, \sigma^2)$. Then is

$$\psi_{N(\mu,\sigma^2)}(t) = e^{it\mu} e^{-\sigma^2 t^2/2}.$$

For $\mu = 0, \sigma^2 = 1$:

$$\frac{d}{dt}\psi_{N(0,1)}(t) = \frac{i}{\sqrt{2\pi}} \int x e^{-x^2/2} e^{itx} dx$$

$$= \frac{i}{\sqrt{2\pi}} \int e^{-x^2/2} it e^{itx} dx = -t\psi_{N(0,1)}(t).$$

A plausible solution of the IVP is $\psi_{N(0,1)}(t) = e^{-t^2/2}$.

- Let $X \sim \exp(\gamma)$. Then is

$$\mathcal{L}_{\exp(\gamma)}(t) = \mathbf{E}[e^{-tX}] = \int_0^\infty \gamma e^{-\gamma x} e^{-tx} dx = \frac{\gamma}{\gamma + t}.$$

Characteristic function and moments

- Proposition 6.14: X real-valued RV.

  If X is in $\mathcal{L}^p$, then $\psi_X \in \mathcal{C}^p(\mathbb{R})$ and for $k = 0, \ldots, p$,

  $$\psi_X^{(k)}(t) = \mathbf{E}[(iX)^k e^{itX}].$$

  In particular, $\psi_X^{(k)}(0) = i^k \mathbf{E}[X^k]$.

  If, specifically, $X \in \mathcal{L}^2$, then

  $\psi_X(t) = 1 + it\mathbf{E}[X] - \frac{t^2}{2}\mathbf{E}[X^2] + \varepsilon(t)t^2$ with $\varepsilon(t) \xrightarrow{t \to 0} 0$.

Proof: $k = 0$ ok; Assume it holds for $k < p$. Then

$$\psi_X^{(k+1)}(t) = \mathbf{E}\left[\frac{d}{dt}(iX)^k e^{itX}\right] = \mathbf{E}[(iX)^{k+1} e^{itX}].$$

Examples: Exponential and normal distribution

- For $X \sim \exp(\gamma)$ is $\mathcal{L}_{\exp(\gamma)}(t) = \gamma/(\gamma + t)$, i.e.

$$\mathbf{E}[X^n] = (-1)^n \frac{d^n}{dt^n} \mathbf{E}[e^{-tX}]\big|_{t=0} = (-1)^n \frac{d^n}{dt^n} \frac{\gamma}{\gamma + t}\Big|_{t=0}$$
$$= \frac{n!\gamma}{(\gamma + t)^{n+1}} \Big| t = 0 = \frac{n!}{\gamma^n}.$$

- For $X \sim N(\mu, \sigma^2)$ is $Xpsi_{N(\mu,\sigma^2)}(t) = e^{it\mu - \sigma^2 t^2/2}$, thus

$$\psi_{N(\mu,\sigma^2)}(t) = 1 + it\mu - \sigma^2 t^2/2 - \mu^2 t^2/2 + \varepsilon(t) t^2$$

with $\varepsilon(t) \xrightarrow{t \to 0} 0$. From this,

$$\mathbf{E}[X] = \mu, \qquad \mathbf{V}[X] = \mathbf{E}[X^2] - \mu^2 = \sigma^2.$$