Albert-Ludwigs Universität Freiburg
Mathematical Institute
Prof. Dr. Peter Pfaffelhuber / Samuel Adeosun

SoSe 2024

Exercises for preparing the exam in Probability Theory

26. Juni 2024

Formal stuff for the exam

- The exam will take place August 5, 2024, at 9:30 in Hörsaal II. The repeat exam will take place in October.
- The time to answer all questions in the exam is 90 minutes. The exam will consist of four exercises.
- Use a non-red pen. Everything written with a pencil cannot be graded, independently from being right or wrong.
- You can use one cheat-sheet (DIN A4, can have text on both sides). Cellphones, smart watches and similar gadgets must be switched off during the exam. Every fraud will be graded „nicht bestanden“ (5,0).
- For computational exercises it is enough to present the result in some form which you might type into a calculator.
- Every exercise must be answered on the sheet below the exercise. If the space is not enough use the paper at the back of the exam. In this case don't forget to indicate which exercise you were working on.
- Every paper that should be graded must have your name on it.

Good luck!

Aufgabe 1 (8 Punkte)

Let $X_1, Y_1, X_2, Y_2, ...$ be real-valued random variables. Show that $((X_n, Y_n))_{n=1,2,...}$ is tight iff both, $(X_n)_{n=1,2,...}$ and $(Y_n)_{n=1,2,...}$ are tight.

Aufgabe 2 (8 Punkte)

Let $X_1, X_2, \dots$ be independent random variables, such that for all k,

$$\mathbb{P}(X_k = 2^k) = 2^{-k} = 1 - \mathbb{P}(X_k = 0), \qquad \text{as well as} \qquad S_n := \sum_{k=1}^{n} X_k.$$

Study $(S_n)_{n\geq 1}$ for almost sure convergence, convergence in probability and weak convergence, as well as $\mathcal{L}^1$ convergence and uniform integrability.

Aufgabe 3 (4+4 Punkte)

Let $t \in \mathbb{R}$ and X be a real-valued random variable. Define $X_t := \min\{X, t\}$.

a) How is $\mathbb{E}[X_t|X]$ defined?

b) Compute $\mathbb{E}[X_t|X]$.

Aufgabe 4 (8 Punkte)

Let $X_1, X_2, \ldots$ iid and real-valued. Show that $\sup_n X_n$ is constant, almost surely.

Aufgabe 5 (8 Punkte)

Let $X_1, \ldots, X_n$ be iid and integrable. Compute $\mathbb{E}[\prod_k X_k | X_1]$.

Aufgabe 6 (10 Punkte)

Let $X_1, X_2, \ldots$ be independent and geo(p) distributed. Furthermore, let p^* be the solution of $pe^p = 1$. Show that: It is $\mathbb{P}(X_n > \frac{\log n}{1-p}$ für infinitely many $n) = 1$ iff $p \leq 1 - p^*$.

Aufgabe 7 (8 Punkte)

Let $U_1, U_2, ... \sim U([0,1])$ independent. Show that $n \min_{1\leq k\leq n} U_k \xrightarrow{n\to\infty} X \sim exp(1)$.

Aufgabe 8 (8 Punkte)

Let $X_1, X_2, \ldots$ real-valued random variables. Show: If $(X_n)_{n=1,2,\ldots}$ is uniformly integrable, then $(X_n)_{n=1,2,\ldots}$ is tight.