universität freiburg

Probability Theory

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https://pfaffelh.github.io/hp/2024ss_wtheorie.html
https://www.stochastik.uni-freiburg.de/

Tutorial 2 - Densities and random variables

Exercise 1 (4 Points)**.**
Let λ, μ and ν be measures on $(\Omega, \mathcal{A})$. Show that:

(a) If for all $\varepsilon > 0$ there exists an $A \in \mathcal{A}$ with $\mu(A) < \varepsilon$ and $\nu(A^c) < \varepsilon$, then $\mu \perp \nu$.

(b) If $\lambda \ll \mu$ and $\mu \perp \nu$, then also $\lambda \perp \nu$.

(c) If $\mu \ll \nu$ and $\mu \perp \nu$, then $\mu \equiv 0$.

Solution.

(a) Let $A_n \in \mathcal{A}$, so that $\mu(A_n) \leq 2^{-n}$ and $\nu(A_n^c) \leq 2^{-n}$ for all n. Then is

$$\mu(\limsup A_n) = \mu\Big(\bigcap_n \bigcup_{m\geq n} A_m\Big) = \lim_{n\to\infty} \mu\Big(\bigcup_{m\geq n} A_m\Big)$$

$$\leq \lim_{n\to\infty} \sum_{m\geq n} \mu(A_m) \leq \lim_{n\to\infty} \sum_{m\geq n} 2^{-m} = \lim_{n\to\infty} 2^{-n+1} = 0$$

and $$\nu\big((\limsup A_n)^c\big) = \nu\Big(\bigcup_n \bigcap_{m\geq n} A_m^c\Big) \leq \sum_{n\geq 1} \underbrace{\nu\Big(\bigcap_{m\geq n} A_m^c\Big)}_{\leq 2^{-m} \text{ for all } m} = 0.$$

(b) Let $A \in \mathcal{A}$ with $\mu(A) = \nu(A^c) = 0$. Because $\lambda \ll \mu$ is then also $\lambda(A) = 0$ and therefore $\lambda \perp \nu$.

(c) If A is as in (b), then with $\mu \ll \nu$ also $\mu(A^c) = 0$ and finally $\mu(\Omega) = \mu(A) + \mu(A^c) = 0$.

Exercise 2 (4 Points)**.**
Let μ and ν be two measures on the measure space $(\Omega, \mathcal{A})$ and let ν be finite. Show that the following statements are equivalent:

(a) $\nu \ll \mu$.

(b) For every $\varepsilon > 0$ there is a $\delta > 0$, such that for all $A \in \mathcal{A}$ with $\mu(A) \leq \delta$, also $\nu(A) \leq \varepsilon$.

Solution.

(b) ⇒ (a): Assume $\nu \not\ll \mu$. Then there is an $A \in \mathcal{A}$ with $\mu(A) = 0 < \varepsilon := \frac{1}{2}\nu(A)$. This means that there is no such δ for A and ε and (b) cannot apply.

(a) ⇒ (b): Assuming (b) does not apply. Then there is a $\varepsilon > 0$ so that for all $n \in \mathbb{N}$ there exists an $A_n \in \mathcal{A}$ with $\mu(A_n) \leq 2^{-n}$ but $\nu(A_n) > \varepsilon$. We set $A_\infty := \limsup_n A_n$ and then obtain, analogue to 1(a), that $\mu(A_\infty) = 0$. However it applies, since ν is finite, that

$$\nu(A_\infty) \geq \limsup_{n\to\infty} \nu(A_n) \geq \varepsilon > 0.$$

Exercise 3 (4 Points)**.**
Give an example of two measures μ,ν with $\nu \ll \mu$ for which there is no density $f\colon \Omega \to \mathbb{R}$ with $d\nu = f d\mu$.

Solution.
A possible example is: let $\Omega \neq \emptyset$ with $\mathcal{F} = \{\emptyset,\Omega\}$. Let μ be a measure with $\mu(\emptyset) = 0$ and $\mu(\Omega) = \infty$. Also ν measure with $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$. Then $\nu \ll \mu$ but there can be no density f since

$$c \cdot \infty \neq 1.$$

Exercise 4 (4 Points)**.**

(a) Give an example of a real-valued random variable $X \neq 0$ with $X \overset{d}{=} -X$.

(b) Show the following: If $X \overset{d}{=} Y$ are E-valued random variables and $f : E \to \mathbb{R}$ is Borel-measurable, then $f(X) \overset{d}{=} f(Y)$.

(c) Show the following: If $X \overset{d}{=} Y$, then $\mathbf{E}[X] = \mathbf{E}[Y]$.

(d) Fill in some details in the proof of Lemma 6.2. ⇐.

Solution.

(a) Consider X such that $X \sim N(0,1)$. The density of X is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Now, let $a \in \mathbb{R}$, we have:

$$\begin{aligned}\mathbb{P}(-X \leq a) = \mathbb{P}(X \geq -a) = \int_{-a}^{\infty} f(x)dx &= -\int_{a}^{-\infty} f(-x)dx \quad \text{(change of variable, } y = -x\text{)} \\ &= \int_{-\infty}^{a} f(x)dx \quad (f \text{ is even}) = \mathbb{P}(X \leq a)\end{aligned}$$

So, $\forall\, a \in \mathbb{R}$, $\mathbb{P}(-X \leq a) = \mathbb{P}(X \leq a)$, that is $X \overset{d}{=} -X$.

Reminder: The standard normal distribution satisfies all the properties of an even function.

(b) Now, we assume $X \overset{d}{=} Y$ and $f : E \to \mathbb{R}$ is Borel-measurable. Note that

$$X, Y : (\Omega,\mathcal{F}) \to (E,\mathcal{F}') \quad f(X), f(Y) : (\Omega,\mathcal{F}) \to (\mathbb{R},\mathcal{B}(\mathbb{R})).$$

To show that $f(X) \overset{d}{=} f(Y)$, we will establish that for all $A \in \mathcal{B}(\mathbb{R})$, $\mathbb{P}(f(X) \in A) = \mathbb{P}(f(Y) \in A)$. Recall that $\mathbb{P}(X \in B) = \mathbb{P}(\{w \in \Omega, X(w) \in B\}) = \mathbb{P}(X^{-1}(B))$. Let $A \in \mathcal{B}\mathbb{R}$, we will therefore show that $\mathbb{P}((f(X))^{-1}(A)) = \mathbb{P}((f(Y))^{-1}(A))$.
Since $f : E \to \mathbb{R}$ is $\mathcal{F}' - \mathcal{B}(\mathbb{R})$ measurable, then, $f^{-1}(A) \in \mathcal{F}'$. Now, $X \overset{d}{=} Y$ means $\mathbb{P}(X^{-1}(f^{-1}(A))) = \mathbb{P}(Y^{-1}(f^{-1}(A)))$ since $f^{-1}(A) \in \mathcal{F}'$. Recall also from Lemma 3.6 that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. Hence,

$$\mathbb{P}((f \circ X)^{-1}(A)) = \mathbb{P}((f \circ Y)^{-1}(A)) \iff \mathbb{P}(f(X) \in A) = \mathbb{P}(f(Y) \in A).$$

(c) If $X \overset{d}{=} Y$, then X and Y have the same distribution function. That is, $F_X(x) = F_Y(x)$. This is because,

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x) = F_Y(x).$$

Now we have that

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x) = \int_{-\infty}^{\infty} x dF_Y(x) = E[Y].$$

(d) Assume there is a there is a $\mathcal{F}'/\mathcal{B}(\overline{\mathbb{R}})$-measurable mapping $\varphi : \Omega' \to \overline{\mathbb{R}}$ with $\varphi \circ X = Z$, let us show that Z is $\sigma(X)$-measurable. That is $\forall\, A \in \mathcal{B}(\overline{\mathbb{R}})$, $Z^{-1}(A) \in \sigma(X)$.
Since φ is $\mathcal{F}'/\mathcal{B}(\overline{\mathbb{R}})$-measurable, then $\varphi^{-1}(A) \in \mathcal{F}'$. X is $\sigma(X)$-measurable, so $X^{-1}(\varphi^{-1}(A)) \in \sigma(X)$. But $Z^{-1} = (\varphi \circ X)^{-1} = X^{-1} \circ \varphi^{-1}$. Hence, $Z^{-1}(A) \in \sigma(X)$.