universität freiburg

Probability Theory **Summer semester 2024**

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https://pfaffelh.github.io/hp/2024ss_wtheorie.html https://www.stochastik.uni-freiburg.de/

Tutorial 2 - Densities and random variables

Exercise 1 (4 Points)**.**
Let λ, μ and ν be measures on $(\Omega,\mathcal{A})$. Show that:

(a) If for all $\varepsilon > 0$ there exists an $A \in \mathcal{A}$ with $\mu(A) < \varepsilon$ and $\nu(A^c) < \varepsilon$, then $\mu \perp \nu$.

(b) If $\lambda \ll \mu$ and $\mu \perp \nu$, then also $\lambda \perp \nu$.

(c) If $\mu \ll \nu$ and $\mu \perp \nu$, then $\mu \equiv 0$.

Exercise 2 (4 Points)**.**
Let μ and ν be two measures on the measure space $(\Omega,\mathcal{A})$ and let ν be finite. Show that the following statements are equivalent:

(a) $\nu \ll \mu$.

(b) For every $\varepsilon > 0$ there is a $\delta > 0$, such that for all $A \in \mathcal{A}$ with $\mu(A) \leq \delta$, also $\nu(A) \leq \varepsilon$.

Exercise 3 (4 Points)**.**
Give an example of two measures μ,ν with $\nu \ll \mu$ for which there is no density $f\colon \Omega \to \mathbb{R}$ with $d\nu = f d\mu$.

Exercise 4 (4 Points)**.**

(a) Give an example of a real-valued random variable $X \neq 0$ with $X \stackrel{d}{=} -X$.

(b) Show the following: If $X \stackrel{d}{=} Y$ are E-valued random variables and $f : E \to \mathbb{R}$ is Borel-measurble, then $f(X) \stackrel{d}{=} f(Y)$.

(c) Show the following: If $X \stackrel{d}{=} Y$, then $\mathbf{E}[X] = \mathbf{E}[Y]$.

(d) Fill in some details in the proof of Lemma 6.2. $\Leftarrow$.